

Logical equivalence for subtyping and recursive types

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March, 2006

Abstract

Subtyping in first order object calculi is studied with respect to the logical semantics obtained by identifying terms that satisfy the same set of predicates, as formalised through an assignment system. It is shown that equality in the full first order ζ -calculus is modelled by this notion, which on turn is included in a Morris style contextual equivalence.

Introduction

Subtyping is a prominent feature of type-theoretic foundation of object oriented programming languages. The basic idea is expressed by subsumption: any piece of code of type A can masquerade as a code of type B whenever A is a subtype of B , written $A <: B$.

In typed calculi, equations can be expressed between terms of the same type; since terms may have several types because of subsumption, it is commonly postulated that if $a = b : A$ (a and b are equal in type A) and $A <: B$ then $a = b : B$ (but not vice-versa): call this *equational subsumption*. In the realm of object calculi, object types are essentially interfaces, and subtyping interface restriction. Therefore subsumption is justified by the intuition that any object which is able to react to messages mentioned in A *a fortiori* will answer correctly to messages in the smaller interfaces represented by its supertypes. Similarly, equational subsumption is understood on the ground of contexts separability: a and b are contextually equivalent at type A if both typeable by A and no context with a hole of type A can separate them. This provides an interpretation of subtyping: $A <: B$ should hold if any pair of terms contextually equivalent at type A cannot be separated at B .

This is semantically understood in two ways, according to the existing literature: either by means of coercions [12], or by inclusion of partial equivalence relations as in [13, 14] and [1] Ch. 14 (see [18] Ch. 10 for a gentle introduction to these approaches). But coercion semantics does not reflect the actual implementation practice of object-oriented languages; on the other hand PER semantics is quite complex to use for reasoning about programs, and suffers technical problems which are still open.

We propose a third approach which, in our view, can lead to a simpler logical framework for reasoning about object oriented programs. It is based on the ideas of logical semantics and domain logic. In the latter perspective, the meaning of a term is determined by the set of the predicates it satisfies, so that two terms are equivalent if they are interpreted by the same set. To account for equivalence “at” a certain type A we relativize this form of absolute indiscernibility to sets of predicates indexed over types, calling them *languages*. Hence a and b are logically equivalent at type A if they satisfy the same set of predicates from the language \mathcal{L}_A associated to A .

We treat three kinds of entities: terms ranged over by a, b, \dots , types ranged over by A, B, \dots and predicates for which we use Greek letters $\sigma, \tau, \phi, \psi, \dots$. We define a system assigning them to each other by means of judgements of the shape $a : A : \sigma$. The system is built in such a way that, if we forget about any one of these three kinds of entities, what remains is still a meaningful assignment system. Let us consider the central rules concerning objects¹ whose syntax is from the ζ -calculus in [1]:

¹These rules are slightly different than those in Section 3, but they are equivalent to them w.r.t. admissibility.

(Val Object) :

$$\frac{\Delta; \Gamma, x_i:A:\sigma_i \vdash_P b_i:B_i : \phi_i \quad (\forall i \in I)}{\Delta; \Gamma \vdash_P [\ell_i = \zeta(x_i^A)b_i \text{ }^{(i \in I)}]:A : \langle \ell_j:\sigma_j \rightarrow \phi_j \rangle} \quad (j \in I)$$

(Val Select) :

$$\frac{\Delta; \Gamma \vdash_P a:A : \langle \ell_j:\sigma \rightarrow \phi \rangle \quad \Delta; \Gamma \vdash_P a:A : \sigma}{\Delta; \Gamma \vdash_P a.\ell_j:B_j : \phi}$$

(Val Update₁) :

$$\frac{\Delta; \Gamma \vdash_P a:A : \sigma \quad \Delta; \Gamma, y:A:\tau \vdash_P b:B_j : \phi}{\Delta; \Gamma \vdash_P (a.\ell_j \Leftarrow \zeta(y^A)b):A : \langle \ell_j:\tau \rightarrow \phi \rangle} \quad (\sigma \neq \omega, j \in I)$$

(Val Update₂) :

$$\frac{\Delta; \Gamma \vdash_P a:A : \langle \ell_j:\phi \rangle \quad \Delta; \Gamma, y:A:\tau \vdash_P b:B_i : \psi}{\Delta; \Gamma \vdash_P (a.\ell_i \Leftarrow \zeta(y^A)b):A : \langle \ell_j:\phi \rangle} \quad (i \neq j)$$

If we forget about predicates we obtain the first order object calculus called $\text{FOb}_{1<:\mu}$ in [1], but for a minor difference (we do not use fold, unfold operators in the term syntax, and consider as isomorphic all the unfoldings of a recursive type).

If instead we erase all types, we get a sort of “intersection type” assignment system for the untyped ζ -calculus (essentially that one used in [15] to characterise the convergence of untyped ζ -terms). Note that the system has features different from the type system: objects are treated as records, which accounts for rules (Val Object) and (Val Update_{*i*}) ($i = 1, 2$) deducing a predicate $\langle \ell:\sigma \rightarrow \phi \rangle$ about a single method labelled by ℓ , that can be put in conjunction (i.e. intersection) with any other similar predicate, possibly with different premise of the arrow. This works because the meaning of $a:\langle \ell:\sigma \rightarrow \phi \rangle$ is not that the self variable has type σ ; rather it claims that σ is a precondition of the method ℓ , which yields a value $a.\ell$ satisfying the post-condition ϕ if σ holds for a (see rule (Val Select) above). Semantically speaking, this is Kamin’s self-application interpretation of objects, accounting for the interpretation of self reference in case of method call and of method overriding. Eventually, this produces the effect that converging terms are characterised by predicates, whereas types that ensure, for example, error free-ness (no “message not understood” error can occur with typed terms at run time), cannot discriminate diverging terms.

We claim that this kind of extended Curry-style assignment system determines a model for the untyped calculus, which is an extension of the filter model for the pure Lambda Calculus of [11]. This depends on the fact that a notion of implication is defined over predicates which is written $\sigma \leq \tau$. this way we build a logic whose filters of formulae provide a denotation to terms, coinciding with the sets of their properties as expressed by the logical formulae (see [4], where this is framed as a form of Stone duality between categories of semi-lattices and of Scott domains).

The third possibility is to forget about terms: we interpret the judgement $A:\sigma$ as: “the predicate σ makes sense of terms of type A ”. The system derives sequents of the form $\Delta \vdash_{\text{PT}} A:\sigma$, where Δ contains assumptions of the shapes $X:\sigma$ or $X:\sigma <: A$; we formally define $\mathcal{L}_A = \{\sigma \mid \emptyset \vdash_{\text{PT}} A:\sigma\}$. This logic of types, inspired to domain logic of [4], is our tool to treat object and recursive types. Consider the rules:

(Type Object), $A \equiv [\ell_i:B_i \text{ }^{(i \in I)}]$:

$$\frac{\Delta \vdash_{\text{PT}} A:\sigma \quad \Delta \vdash_{\text{PT}} B_j:\phi}{\Delta \vdash_{\text{PT}} A:\langle \ell_j:\sigma \rightarrow \phi \rangle} \quad (j \in I)$$

(Type Rec) :

$$\frac{\Delta \vdash_{\text{PT}} \mu X.A:\sigma \quad \Delta, X:\sigma \vdash_{\text{PT}} A:\phi}{\Delta \vdash_{\text{PT}} \mu X.A:\phi}$$

together with the rule that $\Delta \vdash_{\text{PT}} A:\omega$ whenever A is a (well formed) type whose free variables occur in Δ . These rules formalise the concept that both object and recursive types are some kind of fixed point, which is constructed starting with the trivial predicate ω , and iterating the above rules by using the structure of the type expression A . We observe that this definition of the languages of object and recursive types is inductive, and that it makes sense without any consideration about the invariance of the object types nor (and more importantly) about the variance of the occurrences in A of the type variable X within the type $\mu X.A$.

The system induces an equivalence relation $a \simeq b : A$, which, when restricted to closed terms and types, is: $\vdash_P a:A:\sigma$ if and only if $\vdash_P b:B:\sigma$, for all σ . Observing that $\vdash_P a:A:\sigma$ implies $\vdash_{\text{PT}} A:\sigma$, i.e $\sigma \in \mathcal{L}_A$, this formalises the above idea of relativizing logical semantics to types.

To verify that this is a sound theory of the first order object calculus, we prove that $\vdash_{\mathbb{T}} a \leftrightarrow b : A$ (the equational theory of $\text{FOb}_{1<:\mu}$) implies that $a \simeq b : A$, which in turn implies that $a \simeq_A^{\mathcal{O}} b$, namely that a and b cannot be separated by any context $_ : A \vdash C[_] : K$ of any ground type K . This Morris style contextual equivalence is called the *observational equivalence* in [17] and is a maximal consistent theory of the first order object calculus.

To establish the latter results we have to relate our theory of predicates assignment to terms and types to the theory of subtyping. First we establish that if $A <: B$ then $\mathcal{L}_A \supseteq \mathcal{L}_B$: hence the logical theory associated to A is finer than the theory of B , so that any pair of terms which are indiscernible according to A , are such in the coarser theory of B but not vice-versa. This is indeed our interpretation of equational subsumption. Since such inclusions are in general proper, the subtyping rule of the assignment system needs its premise $\Delta \vdash_{\mathbb{P}\mathbb{T}} C : \sigma$:

$$(<:) : \frac{\Delta; \Gamma \vdash_{\mathbb{P}} a : B : \sigma \quad \widehat{\Delta} \vdash_{\mathbb{T}} B <: C \quad \Delta \vdash_{\mathbb{P}\mathbb{T}} C : \sigma}{\Delta; \Gamma \vdash_{\mathbb{P}} a : C : \sigma}$$

(where $\widehat{\Delta}$ is the typing environment obtained from Δ by erasing all predicates). Secondly we find that a converse of rule $(<:)$ also holds: if $\Delta; \Gamma \vdash_{\mathbb{P}} a : B : \sigma$ and $\widehat{\Delta}; \overline{\Gamma} \vdash_{\mathbb{O}} a : A$ for some A such that $\widehat{\Delta} \vdash_{\mathbb{T}} A <: B$, then $\Delta; \Gamma \vdash_{\mathbb{P}} a : A : \sigma$ (Lemma 3.5).

To establish the inclusion of the logical equivalence in the observational equivalence we use a realizability interpretation of predicates instead of types. This is not surprising since the key property of convergence is not captured by the type system, but only by the predicate system, as remarked above. However the logical equivalence is the theory of a model of the typed calculus, which can be constructed as a filter model. We do not enter into the details of this construction in the present paper, and shortly comment on it in Section 7.

The paper is organised as follows: in Section 1 we introduce our variant of the $\text{FOb}_{1<:\mu}$ system. We define predicates, their logic and assignment to types in Section 2. The assignment system, which essentially puts together the typing system with the logic of predicates is defined and studied in Section 3. Basic soundness results of the assignment system with respect to the operational semantics are established via subject reduction and (typed) subject expansion in Section 4. In section 5 we recall the equational theory of $\text{FOb}_{1<:\mu}$ from [1]; then we formally define the logical equivalence and show that the former is included in the latter. Finally in Section 6 we define the observational semantics after [17] and prove the inclusion of the logical semantics in it. We eventually discuss the results presented in the paper, and relate our work to the literature on the subject in Section 7.

1 The first order object calculus

We consider a first order object calculus which is a variant of the calculus called $\text{FOb}_{1<:\mu}$ in [1]. The difference between that version and the one we treat here is that we consider recursive types and their unfoldings as equivalent with respect to subtyping. Consequently, we do not have any syntax to distinguish among the folded and unfolded version of the same term $\text{fold}(A, a)$ and $\text{unfold}(a)$, which will be written simply a .

The calculus is introduced in two steps: we first define pre-types A , pre-terms a and pre-contexts E by simple inductive clauses; these are then refined by means of a typing system. Types and terms are defined together with the type derivation system as well-formed pre-types and well-typed pre-terms respectively.

Definition 1.1 PRE-TYPES, PRE-TERMS, PRE-CONTEXTS. Let \mathcal{K} be a countable set of type constants ranged over by K , and \mathcal{X} a denumerable set of type variables ranged over by X . Let $\mathbb{L} = \{\ell_i \mid i \in \mathbb{N}\}$

Figure 1 Contexts and Types

(Env \emptyset) :	(Type Const) :	(Env X) :	(Type X) :
$\frac{}{\emptyset \vdash_{\top} \diamond}$	$\frac{E \vdash_{\top} \diamond}{E \vdash_{\top} K}$	$\frac{E \vdash_{\top} \diamond}{E, X \vdash_{\top} \diamond} (X \notin \text{dom}(E))$	$\frac{E', X, E'' \vdash_{\top} \diamond}{E', X, E'' \vdash_{\top} X}$
(Type Top) :	(Type Object) :	(Type Arrow) :	(Type Rec) :
$\frac{E \vdash_{\top} \diamond}{E \vdash_{\top} \text{Top}}$	$\frac{E \vdash_{\top} B_i \ (\forall i \in I)}{E \vdash_{\top} [\ell_i : B_i \ (i \in I)]}$	$\frac{E \vdash_{\top} A \quad E \vdash_{\top} B}{E \vdash_{\top} A \rightarrow B}$	$\frac{E, X \vdash_{\top} A}{E \vdash_{\top} \mu X.A}$

be a denumerable set of labels; let \mathcal{C} a countable set of term constants ranged over by c and \mathcal{V} a denumerable set of term variables ranged over by x . The syntax of *pre-types*, *pre-terms* and *pre-contexts* is defined by the following grammar:

$$\begin{aligned}
 \text{Pre-Types:} \quad A, B &::= X \mid K \mid \text{Top} \mid [\ell_i : B_i \ (i \in I)] \mid A \rightarrow B \mid \mu X.A \\
 \text{Pre-Terms:} \quad a, b &::= x \mid c \mid \lambda x^A.a \mid a(b) \mid [\ell_i = \zeta(x_i^A)b_i \ (i \in I)] \mid a.\ell \mid a.\ell \Leftarrow \zeta(x^A)b \\
 \text{Pre-Contexts:} \quad E &::= \emptyset \mid E, X \mid E, X <: A \mid E, x:A
 \end{aligned}$$

where I ranges over finite subsets of \mathbb{N} , and $\ell_i, \ell \in \mathbb{L}$.

The notion of free and bound occurrences of variables, as well as substitution, are defined as usual.

Definition 1.2 FREE AND BOUND, SUBSTITUTION. We say that X is *free* in A if it does not occur within the scope of μX ; it is *bound* otherwise. Similarly, x is *free* in a if it does not occur within the scope of some λx^A nor of some $\zeta(x^A)$; it is *bound* otherwise. We use $\text{fv}(A)$ and $\text{fv}(a)$ to denote the sets of free variables occurring in A and a respectively; $\text{bv}(A)$ and $\text{bv}(a)$ denote the sets of the bound variables.

By $A\{X \leftarrow B\}$ and $a\{x \leftarrow b\}$ we denote the *substitution* of X and x by B and b in A and a respectively, up to renaming of bound variables to avoid variable clashes.

Syntactic equality, up to the renaming of bound variables, is denoted by \equiv .

Definition 1.3 TERMINOLOGY. *i*) An *object pre-term* has the shape $[\ell_i = \zeta(x_i^A)b_i \ (i \in I)]$;
ii) in $\ell_i = \zeta(x_i^A)b_i$, the variable x_i^A is the *self* variable, and b_i is the *body* of the *method* ℓ_i ; if $x_i \notin \text{fv}(b_i)$ then ℓ_i is more properly seen as a *field*;
iii) $a.\ell$ denotes the *invocation* of method ℓ of the object a , if a evaluates to an object having such a method;
iv) $a.\ell \Leftarrow \zeta(x^A)b$ is the *overriding* of method ℓ in the object to which a evaluates, if any;
v) Functional abstraction and application are represented as usual in typed λ -calculi, via $\lambda x^A.a$ and $a(b)$;
vi) A pre-type of the shape $[\ell_i : B_i \ (i \in I)]$ will be used for an object whose methods ℓ_i have *return type* B_i ;
vii) $A \rightarrow B$ is the usual *functional* type and $\mu X.A$ is a *recursive* type;
viii) Top is the *maximal* type w.r.t. the subtyping relation $<:$.

We will now repeat the construction of the definition of type assignment. The notions of *inference rule* and *derivation* are as usual. As in [1], we will use a short-hand for rules, and write for example (where $I = \{1, \dots, n\}$)

$$\frac{E, x_i:A \vdash_{\circ} b_i : B_i \quad (\forall i \in I)}{E \vdash_{\circ} [\ell_i = \varsigma(x_i^A) b_i \quad (i \in I)] : A} \quad \text{for} \quad \frac{E, x_1:A \vdash_{\circ} b_1 : B_1 \quad \dots \quad E, x_n:A \vdash_{\circ} b_n : B_n}{E \vdash_{\circ} [\ell_i = \varsigma(x_i^A) b_i \quad (i \in I)] : A}$$

In what follows, we will present a number of type assignment systems that are interdependent. In fact, we will introduce the following systems, with the added intended meaning:

$$\begin{array}{ll} E \vdash_{\top} \diamond & E \text{ is a well formed context (shortly a context)} \\ E \vdash_{\top} A & A \text{ is a type within the context of } E; \\ E \vdash_{\top} A <: B & A \text{ is a subtype of } B \text{ within the context of } E \\ E \vdash_{\circ} a : A & a \text{ has type } A \text{ in the context } E; \end{array}$$

These notions will be introduced in sequence.

Definition 1.4 PRE-CONTEXT. A *pre-context* E is defined via the grammar:

$$E ::= \emptyset \mid E, X \mid E, X <: A \mid E, x:A$$

where X is a type variable, A a pre-type, x a term variable.

The *domain* of a pre-context E is the set of type and term variables occurring in E , and is defined by:

$$\begin{aligned} \text{dom}(\emptyset) &= \emptyset, \\ \text{dom}(E, X) &= \text{dom}(E, X <: A) = \text{dom}(E) \cup \{X\}, \\ \text{dom}(E, x:A) &= \text{dom}(E) \cup \{x\}. \end{aligned}$$

Although pre-contexts are formally sequences, by abuse of notation we shall treat them as sets and write $X \in E$, $X <: A \in E$ or $x:A \in E$ to mean that $X, X <: A$ and $x:A$ occur as elements of the sequence E respectively.

Definition 1.5 JUDGEMENTS AND SEQUENTS. A *type judgement* has one of the forms \diamond , A , $A <: B$, or $a : A$, where A and B are pre-types and a is a pre-term. A *type sequent* has the form $E \vdash \Theta$ where E is a pre-environment and Θ is a type judgement.

Definition 1.6 THE TYPED OBJECT CALCULUS. A sequent $E \vdash \Theta$ is *derivable* in the calculus of objects if there exists a derivation whose inferences are instances of the rules in figures 1, 2 and 3, such that $E \vdash \Theta$ appears in the bottom line.

We say that E is a *context*, A is a *type* in the context E and a a *term* (of type A) in the context E , if $E \vdash_{\top} \diamond$, $E \vdash_{\top} A$ and $E \vdash_{\circ} a : A$ are derivable respectively.

We will write $\mathcal{D} :: E \vdash \Theta$ when \mathcal{D} is a derivation whose conclusion is the sequent $E \vdash \Theta$, and will write $E \vdash \Theta$ if there exists a derivation \mathcal{D} such that $\mathcal{D} :: E \vdash \Theta$, i.e. this sequent is derivable.

A sequent $E \vdash \Theta$ formalises the idea that Θ is a consequence of the judgements in E . In particular,

- if $\Theta \equiv \diamond$ then the intended meaning is that E is a *well-formed context*;
- if $\Theta \equiv A$ then A is a *well-formed type*, whose free type variables have been declared in the context E ;
- if $\Theta \equiv A <: B$ then A is a subtype of B under the subtyping assumptions in E ; and
- if $\Theta \equiv a:A$ then a is a *well-formed term* of type A under the assumption that all its free variables have been given a type in the context E .

From now on, when dealing with contexts, types and terms, we will assume they are well formed.

Figure 2 Subtyping

$$\begin{array}{c}
 \text{(Sub Refl) :} \quad \frac{E \vdash_{\top} A}{E \vdash_{\top} A <: A} \quad \text{(Sub Top) :} \quad \frac{E \vdash_{\top} A}{E \vdash_{\top} A <: \text{Top}} \quad \text{(Sub Trans) :} \quad \frac{E \vdash_{\top} A <: B \quad E \vdash_{\top} B <: C}{E \vdash_{\top} A <: C} \\
 \\
 \text{(Sub X) :} \quad \frac{E', X <: A, E'' \vdash_{\top} \diamond}{E', X <: A, E'' \vdash_{\top} X <: A} \quad \text{(Env X <:) :} \quad \frac{E \vdash_{\top} A}{E, X <: A \vdash_{\top} \diamond} \quad (X \notin \text{dom}(E)) \quad \text{(Type X <:) :} \quad \frac{E', X <: A, E'' \vdash_{\top} \diamond}{E', X <: A, E'' \vdash_{\top} X} \\
 \\
 \text{(Sub Object) :} \quad \frac{E \vdash_{\top} B_i \quad (\forall i \in I)}{E \vdash_{\top} [\ell_i : B_i \text{ }^{i \in I}] <: [\ell_i : B_i \text{ }^{i \in J}]} \quad (J \subseteq I) \quad \text{(Sub Arrow) :} \quad \frac{E \vdash_{\top} A' <: A \quad E \vdash_{\top} B <: B'}{E \vdash_{\top} A \rightarrow B <: A' \rightarrow B'} \\
 \\
 \text{(Sub Rec}_1\text{) :} \quad \frac{E \vdash_{\top} \mu X.A \quad E \vdash_{\top} A\{X \leftarrow \mu X.A\}}{E \vdash_{\top} \mu X.A <: A\{X \leftarrow \mu X.A\}} \quad \text{(Sub Rec}_2\text{) :} \quad \frac{E \vdash_{\top} \mu X.A \quad E \vdash_{\top} A\{X \leftarrow \mu X.A\}}{E \vdash_{\top} A\{X \leftarrow \mu X.A\} <: \mu X.A} \\
 \\
 \text{(Sub Rec}_3\text{) :} \quad \frac{E \vdash_{\top} \mu X.A \quad E \vdash_{\top} \mu Y.B \quad E, Y, X <: Y \vdash_{\top} A <: B}{E \vdash_{\top} \mu X.A <: \mu Y.B}
 \end{array}$$

- Lemma 1.7*
- i) If $E \vdash_{\top} A$ and $X \notin \text{dom}(E)$ then $E \vdash_{\top} A =: \mu X.A$,
 - ii) if $E \vdash_{\top} K <: A$ for ground K , then $E \vdash_{\top} A =: K$,
 - iii) if $E \vdash_{\top} A <: B \rightarrow C$ then there exist A', A'' such that $E \vdash_{\top} A =: A' \rightarrow A''$ and $E \vdash_{\top} B <: A'$ and $E \vdash_{\top} A'' <: C$,
 - iv) if $E \vdash_{\top} A <: [\ell_i : B_i \text{ }^{(i \in I)}]$ then there exists some $J \supseteq I$ such that $\vdash_{\top} A =: [\ell_j : B_j \text{ }^{(j \in J)}]$.

Proof: Immediate by inspection of rules. Observe that if $E \vdash_{\top} A$ and $X \notin \text{dom}(E)$ then $X \notin \text{fv}(A)$; hence $E \vdash_{\top} \mu X.A =: A$ by (Sub Rec₁), (Sub Rec₂), since $A \equiv A\{X \leftarrow \mu X.A\}$ in this case. ■

Lemma 1.8 TYPE GENERATION LEMMA. i) If $E \vdash_{\top} A$ then $E \vdash_{\top} \diamond$;

- ii) if $E, X \vdash_{\top} B$, then $X \notin \text{dom}(E)$;
- iii) if $E \vdash_{\top} X$ then $X \in \text{dom}(E)$;
- iv) if $E \vdash_{\top} A$ and $X \in \text{fv}(A)$ then $X \in \text{dom}(E)$;
- v) if $E \vdash_{\top} A <: B$ then both $E \vdash_{\top} A$ and $E \vdash_{\top} B$;
- vi) if $E', X <: A, E'' \vdash_{\top} \diamond$ then $E' \vdash_{\top} A$;
- vii) if $E \vdash_{\top} [\ell_i : B_i \text{ }^{(i \in I)}]$ then $E \vdash_{\top} B_i$ for all $i \in I$;
- viii) if $E \vdash_{\top} A \rightarrow B$ then both $E \vdash_{\top} A$ and $E \vdash_{\top} B$;
- ix) if $E \vdash_{\top} \mu X.A$ then $E, X \vdash_{\top} A$;
- x) if $E \vdash_{\top} [\ell_i : B_i \text{ }^{(i \in I)}]$ then $E \vdash_{\top} B_i$ for all $i \in I$.

Proof: Straightforward. ■

The following is easy to show:

Lemma 1.9 TYPED TERMS GENERATION LEMMA. i) if $E \vdash_0 a : A$ then $E \vdash_{\top} A$;

- ii) if $E \vdash_0 a : A$ and $x \in \text{fv}(a)$ then $x \in \text{dom}(E)$;
- iii) If $E \vdash_0 c : C$ and c is a constant of some ground type K , then $C \equiv K$.

Figure 3 Typed terms

$$\begin{array}{c}
 \text{(Val Const) :} \\
 \frac{E \vdash_{\top} \diamond}{E \vdash_{\circ} c : K} \text{ (for } c \text{ of type } K)
 \end{array}
 \qquad
 \begin{array}{c}
 \text{(Env } x) : \\
 \frac{E \vdash_{\top} A}{E, x:A \vdash_{\circ} \diamond} \text{ (} x \notin \text{dom}(E)\text{)}
 \end{array}
 \qquad
 \begin{array}{c}
 \text{(Val } x) : \\
 \frac{E', x:A, E'' \vdash_{\top} \diamond}{E', x:A, E'' \vdash_{\circ} x : A}
 \end{array}$$

$$\begin{array}{c}
 \text{(Val Fun) :} \\
 \frac{E, x:A \vdash_{\circ} a : B}{E \vdash_{\circ} \lambda x^A. a : A \rightarrow B}
 \end{array}
 \qquad
 \begin{array}{c}
 \text{(Val Appl) :} \\
 \frac{E \vdash_{\circ} a : A \rightarrow B \quad E \vdash_{\circ} b : A}{E \vdash_{\circ} a(b) : B}
 \end{array}
 \qquad
 \begin{array}{c}
 \text{(Val Subsumption) :} \\
 \frac{E \vdash_{\circ} a : A \quad E \vdash_{\top} A <: B}{E \vdash_{\circ} a : B}
 \end{array}$$

In the subsequent rules, let $A \equiv [\ell_i : B_i \text{ }^{i \in I}]$:

$$\begin{array}{c}
 \text{(Val Object) :} \\
 \frac{E, x_i : A \vdash_{\circ} b_i : B_i \quad (\forall i \in I)}{E \vdash_{\circ} [\ell_i = \varsigma(x_i^A) b_i \text{ }^{(i \in I)}] : A}
 \end{array}
 \qquad
 \begin{array}{c}
 \text{(Val Select) :} \\
 \frac{E \vdash_{\circ} a : A}{E \vdash_{\circ} a. \ell_j : B_j} \text{ (} j \in J\text{)}
 \end{array}
 \qquad
 \begin{array}{c}
 \text{(Val Update) :} \\
 \frac{E \vdash_{\circ} a : A \quad E, x:A \vdash_{\circ} b : B_j \text{ (} j \in J\text{)}}{E \vdash_{\circ} (a. \ell_j \Leftarrow \varsigma(x^A) b) : A}
 \end{array}$$

- iv) If $E \vdash_{\circ} x : C$ then there exists an A such that $x:A$ occurs in E and $E \vdash_{\top} A <: C$.
- v) If $E \vdash_{\circ} [\ell_i = \varsigma(x_i^A) b_i \text{ }^{(i \in I)}] : C$, then there exist $A \equiv [\ell_i : B_i \text{ }^{(i \in I)}]$ such that $E \vdash_{\top} A <: C$ and, for all $i \in I$, $A \equiv A_i$ and $E, x_i : A \vdash_{\circ} b_i : B_i$.
- vi) If $E \vdash_{\circ} a. \ell : C$ then there exists an $A \equiv [\dots, \ell : B, \dots]$ such that $E \vdash_{\top} A$, $E \vdash_{\top} B <: C$ and $E \vdash_{\circ} a : A$.
- vii) If $E \vdash_{\circ} (a. \ell \Leftarrow \varsigma(x^D) b) : C$ then $E \vdash_{\top} D <: C$ and there exists an $A \equiv [\dots, \ell : B, \dots]$ such that $E \vdash_{\top} A <: D$, $E \vdash_{\circ} a : A$ and $E, x:D \vdash_{\circ} b : B$.
- viii) If $E \vdash_{\circ} \lambda x^A. a : C$, then there exists B such that $E, x:A \vdash_{\circ} a : B$, and $E \vdash_{\top} A \rightarrow B <: C$.
- ix) If $E \vdash_{\circ} a(b) : C$, then there exist A, B such that $E \vdash_{\circ} a : A \rightarrow B$, $E \vdash_{\circ} b : A$ and $E \vdash_{\top} B <: C$.

Proof: By induction on the structure of derivations. ■

The following property is standard, and allows to generalise derivable results.

- Lemma 1.10** WEAKENING. i) if $E \vdash_{\top} A$ and $X \notin \text{dom}(E)$, then $E, X \vdash_{\top} A$; if also $E \vdash_{\top} C$ then $E, X <: C \vdash_{\top} A$;
- ii) if $E \vdash_{\top} A <: B$ and $X \notin \text{dom}(E)$, then $E, X \vdash_{\top} A <: B$ if also $E \vdash_{\top} C$ then $E, X <: C \vdash_{\top} A <: B$;
- iii) if $E \vdash_{\circ} a : A$, $E \vdash_{\top} B$ and $x \notin \text{dom}(E)$, then $E, x:B \vdash_{\circ} a : A$.

Proof: By induction on derivations. We just remark that, e.g. in case of (i), if $E \vdash_{\top} A$ then $E \vdash_{\top} \diamond$ by Lemma 1.8, so that $E, X \vdash_{\top} \diamond$ because $X \notin \text{dom}(E)$ is the side condition of rule (Env X). It follows that the derivation of $E, X \vdash_{\top} A$ is essentially the same as the given derivation of $E \vdash_{\top} A$, but for the sub-derivation of the sequent $E, X \vdash_{\top} \diamond$. In case of $E, X <: C \vdash_{\top} A$ we need the hypothesis $E \vdash_{\top} C$ because of the premise of rule (Env X<:).

The proofs of the other items are similar. ■

2 Typed Predicates and Languages

In this section we will introduce the syntax of the predicates and an assignment system to syntactically derive judgements associating predicates to types under the assumption of similar judgements about a finite set of type variables.

Term properties are formalised by predicates, which in turn are classified by types. Predicates are transparently intersection types for a λ -calculus with records, and come from [15]. The essential difference is that the set of predicates is stratified into languages (see [16, 9]), in such a way that whenever a predicate can be deduced for a (closed) term a , it belongs to the language \mathcal{L}_A associated with the (closed) type A .

Much in the style of [8], in this section we will present a notion of *strict intersection types*, called *strict predicates* here. Using these, we will define in the next section a notion of *predicate assignment*, which will consist basically of associating a predicate to a typed term.

Definition 2.1 PREDICATES. The set \mathcal{P} of *predicates*, ranged over by σ, τ, \dots and its subset \mathcal{P}_s of *strict predicates* ranged over by ϕ, ψ, \dots , are defined through the grammar:

$$\begin{aligned}\phi &::= \kappa \mid \omega \mid (\sigma \rightarrow \phi) \mid \langle \ell : \phi \rangle \\ \sigma, \tau &::= \phi \mid (\sigma \wedge \tau)\end{aligned}$$

where κ ranges over a countable set of atoms, and $\ell \in \mathbb{L}$ is any (method) label.

Since \wedge is commutative and associative w.r.t. the equivalence introduced below in Definition 2.2, we omit brackets and write $\bigwedge_{i \in I} \sigma_i$ for $\sigma_1 \wedge \dots \wedge \sigma_n$. Also, rather than $\langle \ell_1 : \phi_1 \rangle \wedge \dots \wedge \langle \ell_n : \phi_n \rangle$ where the ℓ_i are pair-wise distinct, we will write $\langle \ell_i : \phi_i \rangle^{(i \in I)}$. By definition, any predicate $\sigma \in \mathcal{P}$ is such that $\sigma \equiv \bigwedge_{i \in I} \phi_i$ for some non empty finite I and certain $\phi_i \in \mathcal{P}_s$. We shall use ϕ, ψ possibly with apices and indexes for elements of \mathcal{P}_s , while $\sigma, \tau \in \mathcal{P} \supseteq \mathcal{P}_s$ (with similar decorations) may be strict or not.

Predicates are strict intersection types in the sense of [7, 8] but for the fact that the type constant ω is no longer treated as the empty intersection, which allows for occurrences of ω at the right of arrows. Also record predicates $\langle \ell : \phi \rangle$ are added. With respect to ordinary intersection types, which have been introduced by several authors in a series of papers (see e.g. [8] for references), the occurrence of intersection \wedge is not allowed at the right of an arrow; this is a technical choice and a departure from [9], making the proof theory of the system more manageable, since it allows for a more syntax directed treatment of the assignment system, without loss in the expressivity.

Atomic predicates κ are intended to describe elements of atomic types in the domain of interpretation, but they can be used also to denote subsets of such values (e.g. the odd or even integers in the interpretation of *Int*), allowing for a limited form of abstract interpretation. $\sigma \rightarrow \phi$ is the property of functions sending elements satisfying σ into elements satisfying ϕ . $\langle \ell : \phi \rangle$ is the property of records having values that satisfy ϕ associated with the field ℓ . Predicates ω and $\sigma \wedge \tau$ mean ‘*truth*’ and ‘*conjunction*’ respectively. It should be noted that arbitrary conjunctive predicates like $(\sigma \rightarrow \phi) \wedge \langle \ell : \psi \rangle$ are allowed by the above definition, although never derived for any type nor for any term by the assignment systems.

To build a logic of predicates we need a notion of implication, written $\sigma \leq \tau$ (read as: “ σ implies τ ”), which is a reflexive and transitive relation on predicates, as defined below. Also, as in [8], but differently w.r.t. [7], we define \leq to be *contra-variant* in arrow types.

Definition 2.2 PREDICATE PRE-ORDER. The relation \leq over predicates is defined as the last pre-order such that for any $\sigma, \tau \in \mathcal{P}$ and $\phi, \psi \in \mathcal{P}_s$:

- i) $\sigma \leq \omega$,
- ii) $(\sigma \wedge \tau)$ is the meet of σ and τ ,
- iii) $(\sigma \rightarrow \omega) \leq (\omega \rightarrow \omega)$,
- iv) $\sigma \geq \tau, \phi \leq \psi \Rightarrow (\sigma \rightarrow \phi) \leq (\tau \rightarrow \psi)$,
- v) $\phi \leq \psi \Rightarrow \langle \ell : \phi \rangle \leq \langle \ell : \psi \rangle$, for any $\ell \in \mathbb{L}$.

Finally $\sigma = \tau \iff \sigma \leq \tau \leq \sigma$, and we write $\sigma < \tau$ if $\sigma \leq \tau$ and $\sigma \neq \tau$. A predicate is *trivial* if it is equivalent to ω .

The relation \leq differs from that considered in [11], in that there $\omega \rightarrow \omega = \omega$, whereas here we only allow $\omega \rightarrow \omega < \omega$. This is natural in the present context of lazy evaluation, where an abstraction should always have a conjunction of arrow predicates, hence different from ω , even if it does not return a result. Since strict intersection types are essentially representatives of equivalence classes of type in [11], in [8], $\omega \rightarrow \omega$ is not a type; any term typeable by that type in [11] is typeable only by ω in [8].

Lemma 2.3 For any $\sigma, \tau \in \mathcal{P}$, and $\phi, \psi \in \mathcal{P}_s$:

- i) σ is trivial if and only if $\sigma \equiv \bigwedge_{i \in I} \omega$ for any (finite) I ,
- ii) $\sigma \rightarrow \phi \leq \omega \rightarrow \omega$ and $\langle \ell : \phi \rangle \leq \langle \ell : \omega \rangle$ for any $\ell \in \mathbb{L}$,
- iii) $I \supseteq J, \forall j \in J. \phi_j \leq \psi_j \Rightarrow \langle \ell_i : \phi_i^{i \in I} \rangle \leq \langle \ell_j : \psi_j^{j \in J} \rangle$.

Proof: Immediate consequences of Definition 2.2. In particular, by inspection of the axioms of \leq it is evident that if $\omega \not\equiv \phi \in \mathcal{P}_s$ then $\phi \neq \omega$; hence (i) follows by the fact that $\sigma \equiv \bigwedge_{i \in I} \phi_i$ for some $\phi_i \in \mathcal{P}_s$ and that if either $\sigma_i \neq \omega$ for $i = 1, 2$, then $\sigma_1 \wedge \sigma_2 \neq \omega$ for any σ_i .

About (iii) we note that $\langle \ell_i : \phi_i^{i \in I} \rangle \equiv \bigwedge_{i \in I} \langle \ell_i : \phi_i \rangle$, and that the meet operation \wedge is monotonic w.r.t. \leq . ■

The first part of this lemma implies that to be a trivial predicate is decidable. The subsequent part says that $\omega \rightarrow \omega$ and $\langle \ell : \omega \rangle$ are the largest non trivial predicates among arrow and record predicates (with a certain label ℓ) respectively. The last part claims that, with respect to \leq , record predicates mirror record subtyping in width and in depth.

- Lemma 2.4**
- i) If $\sigma \leq \tau \rightarrow \phi$, then there are I and σ_i, ψ_i for every $i \in I$, such that $\sigma = \bigwedge_{i \in I} (\sigma_i \rightarrow \psi_i)$ and there exists $J \subseteq I$ such that both $\bigwedge_{j \in J} \sigma_j \geq \tau$ and $\bigwedge_{j \in J} \psi_j \leq \phi$,
 - ii) if $\sigma \leq \langle \ell_j : \psi_j^{j \in J} \rangle$ then there exists $I \supseteq J$ and ϕ_i for every $i \in I$, such that $\sigma = \langle \ell_i : \phi_i^{i \in I} \rangle$ and $\phi_j \leq \psi_j$ for all $j \in J$.
 - iii) For all $\sigma, \tau, \sigma \leq \tau$ if and only if there are $\sigma_i (i \in I), \tau_j (j \in J)$ such that $\sigma = \bigwedge_{i \in I} \sigma_i, \tau = \bigwedge_{j \in J} \tau_j$, and, for every $j \in J$, there is an $i \in I$ such that $\sigma_i \leq \tau_j$.

Proof: By induction on the definition of \leq (2.2). Note that the statements would become false with \equiv in place of $=$ since equation $\sigma \wedge \omega = \sigma$ trivially holds for any σ . ■

Definition 2.5 PREDICATES CONTEXTS. i) *Predicate pre-contexts* are defined by the following grammar:

$$\Delta ::= \emptyset \mid \Delta, X : \sigma \mid \Delta, X : \sigma < : A$$

where X is a type variable, σ is a predicate, and A is a pre-type.

- ii) By $\widehat{\Delta}$ we denote the pre-context E obtained from Δ by erasing all predicates:

$$\begin{aligned} \widehat{\emptyset} &\equiv \emptyset, \\ \widehat{\Delta, X : \sigma} &\equiv \widehat{\Delta}, X, \\ \widehat{\Delta, X : \sigma < : A} &\equiv \widehat{\Delta}, X < : A. \end{aligned}$$

Set $dom(\Delta) = dom(\widehat{\Delta})$.

- iii) A *predicate context* is a predicate pre-context Δ such that $\Delta \vdash_{PT} \diamond$ is derivable in the system of Figure 2, which we call the *predicates to types assignment system*.
- iv) We extend the relation \leq as defined on predicates to predicate contexts by: $\Delta \leq \Delta'$ if and only if, for every $X : \sigma' \in \Delta'$ or $X : \sigma' < : A \in \Delta'$ there exists $X : \sigma \in \Delta$ or $X : \sigma < : A \in \Delta$ respectively, such that $\sigma \leq \sigma'$.

Figure 4 Predicates to Types Assignment System

$$\begin{array}{c}
 \text{(Env } \emptyset \text{)} : \quad \frac{}{\emptyset \vdash_{\mathbb{P}\mathbb{T}} \diamond} \quad \text{(Env } X \text{)} : \quad \frac{\Delta \vdash_{\mathbb{P}\mathbb{T}} \diamond}{\Delta, X:\sigma \vdash_{\mathbb{P}\mathbb{T}} \diamond} (X \notin \text{dom}(\Delta)) \quad \text{(Env } X < \text{)} : \quad \frac{\Delta \vdash_{\mathbb{P}\mathbb{T}} A:\tau}{\Delta, X:\sigma <: A \vdash_{\mathbb{P}\mathbb{T}} \diamond} (X \notin \text{dom}(\Delta), \sigma \leq \tau) \\
 \\
 \text{(Type Const)} : \quad \frac{\Delta \vdash_{\mathbb{P}\mathbb{T}} \diamond}{\Delta \vdash_{\mathbb{P}\mathbb{T}} K:\kappa} \quad \text{(Type } X \text{)} : \quad \frac{\Delta', X:\sigma, \Delta'' \vdash_{\mathbb{P}\mathbb{T}} \diamond}{\Delta', X:\sigma, \Delta'' \vdash_{\mathbb{P}\mathbb{T}} X:\phi} (\sigma \leq \phi) \quad \text{(Type } X < \text{)} : \quad \frac{\Delta', X:\sigma <: A, \Delta'' \vdash_{\mathbb{P}\mathbb{T}} \diamond}{\Delta', X:\sigma <: A, \Delta'' \vdash_{\mathbb{P}\mathbb{T}} X:\phi} (\sigma \leq \phi) \\
 \\
 \text{(Type Arrow)} : \quad \frac{\Delta \vdash_{\mathbb{P}\mathbb{T}} A:\sigma \quad \Delta \vdash_{\mathbb{P}\mathbb{T}} B:\phi}{\Delta \vdash_{\mathbb{P}\mathbb{T}} A \rightarrow B:\sigma \rightarrow \phi} \quad \text{(Type Object), } A \equiv [\ell_i:B_i \text{ } (i \in I)] : \quad \frac{\Delta \vdash_{\mathbb{P}\mathbb{T}} A:\sigma \quad \Delta \vdash_{\mathbb{P}\mathbb{T}} B_j:\phi}{\Delta \vdash_{\mathbb{P}\mathbb{T}} A:\langle \ell_j:\sigma \rightarrow \phi \rangle} (j \in I) \\
 \\
 \text{(Type Rec)} : \quad \frac{\Delta \vdash_{\mathbb{P}\mathbb{T}} \mu X.A:\sigma \quad \Delta, X:\sigma \vdash_{\mathbb{P}\mathbb{T}} A:\phi}{\Delta \vdash_{\mathbb{P}\mathbb{T}} \mu X.A:\phi} \quad (\omega) : \quad \frac{\Delta \vdash_{\mathbb{P}\mathbb{T}} \diamond \quad \widehat{\Delta} \vdash_{\mathbb{T}} A}{\Delta \vdash_{\mathbb{P}\mathbb{T}} A:\omega} \quad (\wedge) : \quad \frac{\Delta \vdash_{\mathbb{P}\mathbb{T}} A:\phi_i \quad (\forall i \in I)}{\Delta \vdash_{\mathbb{P}\mathbb{T}} A:\bigwedge_{i \in I} \phi_i}
 \end{array}$$

Definition 2.6 ASSIGNMENT OF PREDICATES TO TYPES: LANGUAGES. Let Δ be a predicate context, A a type and σ a predicate. We write $\Delta \vdash_{\mathbb{P}\mathbb{T}} A:\sigma$ if this statement can be derived using the rules of Figure 2.

Given a closed type A we define the *language* of A as the set $\mathcal{L}_A = \{\sigma \in \mathcal{P} \mid \emptyset \vdash_{\mathbb{P}\mathbb{T}} A:\sigma\}$.

Notice that, in the definition of the system as in Figure 2, $\sigma, \tau \in \mathcal{P}$ while $\phi \in \mathcal{P}_S$: otherwise one could assign to types predicates which are not in \mathcal{P} at all.

As it is apparent from Definition 2.6, we are essentially interested into closed types and their languages. The reason why we introduce contexts Δ in the system is for a proper handling of assumptions of predicates assigned to type variables, which may occur in recursive types. This parallels the usage of contexts E in the type system for $\vdash_{\mathbb{T}}$.

The logical interpretation of types we are proposing is as collections of predicates, closed under conjunction (by rule (\wedge)) and logical implication (by the admissibility of rule (\leq) in Lemma 2.14): that is types are seen as propositional theories.

Since predicates are properties of terms, which in turn are polymorphic typed entities, the soundness criterion we have in mind for the judgements $A:\sigma$ is that there exists some term of type A satisfying σ : this is what we mean by saying that σ makes sense of entities of type A . We observe that this is the very departure of the present work from the endogenous logic of [4]: while there the logical interpretation of polymorphism is not considered (but for the limited case of recursive types), so that the logical and denotational interpretations are pairwise disjoint for non isomorphic types, the present construction allows for proper inclusions and non empty intersections of languages.

Example 2.7 Let $A \equiv [\ell_0:\text{Int}, \ell_1:\text{Int}]$, and suppose that $\mathbf{O}, \mathbf{E} \in \mathcal{L}_{\text{Int}}$ are the predicates of being odd and even integer respectively. Then we can derive $\vdash_{\mathbb{P}\mathbb{T}} A:\langle \ell_0:\omega \rightarrow \mathbf{O} \rangle$ as follows (by omitting some obvious inferences):

$$\frac{\frac{\emptyset \vdash_{\mathbb{P}\mathbb{T}} \diamond \quad \vdash_{\mathbb{T}} A}{\vdash_{\mathbb{P}\mathbb{T}} A:\omega} (\omega)}{\vdash_{\mathbb{P}\mathbb{T}} A:\langle \ell_0:\omega \rightarrow \mathbf{O} \rangle} \vdash_{\mathbb{P}\mathbb{T}} \text{Int}:\mathbf{O} \quad \text{(Type Object)}$$

Once this is given we can derive more complex statements like:

$$\frac{\frac{\frac{\vdash_{\text{PT}} A : \langle \ell_0 : \omega \rightarrow \mathbf{E} \rangle}{\vdash_{\text{PT}} A : \langle \ell_0 : \omega \rightarrow \mathbf{E} \rangle} \quad \frac{\frac{\vdash_{\text{PT}} A : \langle \ell_0 : \omega \rightarrow \mathbf{O} \rangle \quad \vdash_{\text{PT}} \text{Int} : \mathbf{O}}{\vdash_{\text{PT}} A : \langle \ell_1 : \langle \ell_0 : \omega \rightarrow \mathbf{O} \rangle \rightarrow \mathbf{O} \rangle} \text{(Type Object)}}{\vdash_{\text{PT}} A : \langle \ell_1 : \langle \ell_0 : \omega \rightarrow \mathbf{O} \rangle \rightarrow \mathbf{O} \rangle} \text{(\wedge)}}{A \vdash_{\text{PT}} \langle \ell_0 : \omega \rightarrow \mathbf{E}, \ell_1 : \langle \ell_0 : \omega \rightarrow \mathbf{O} \rangle \rightarrow \mathbf{O} \rangle}$$

where the derivation of $\vdash_{\text{PT}} A : \langle \ell_0 : \omega \rightarrow \mathbf{E} \rangle$ is similar to that of $\vdash_{\text{PT}} A : \langle \ell_0 : \omega \rightarrow \mathbf{O} \rangle$.

The predicate $\langle \ell_0 : \omega \rightarrow \mathbf{E}, \ell_1 : \langle \ell_0 : \omega \rightarrow \mathbf{O} \rangle \rightarrow \mathbf{O} \rangle$ is satisfied by any object having at least methods labelled by ℓ_0 and ℓ_1 (that we identify with the respective methods), where ℓ_0 returns an even number, no matter which is the actual state of the object; ℓ_1 returns an odd integer provided that ℓ_0 does. This makes sense, however, and so it is not contradictory, since the second conjunct of the predicate is only a conditional. Moreover this is essential for handling method overriding: see Example 4.8.

Concerning recursive types, consider the following derivation:

$$\frac{\frac{\frac{\vdash_{\text{T}} \mu X. X \rightarrow X}{\vdash_{\text{PT}} \mu X. X \rightarrow X : \omega} (\omega) \quad \frac{X : \omega \vdash_{\text{PT}} X : \omega \quad X : \omega \vdash_{\text{PT}} X : \omega}{X : \omega \vdash_{\text{PT}} X \rightarrow X : \omega \rightarrow \omega} \text{(Type Arrow)}}{\vdash_{\text{PT}} \mu X. X \rightarrow X : \omega \rightarrow \omega} \text{(Type Rec)}}$$

From this it is then not difficult to see that $(\omega \rightarrow \omega) \rightarrow (\omega \rightarrow \omega) \in \mathcal{L}_{\mu X. X \rightarrow X}$; but also the unbalanced unfolds (on the predicate side) $(\omega \rightarrow \omega) \rightarrow \omega$ and $\omega \rightarrow (\omega \rightarrow \omega)$ are in $\mathcal{L}_{\mu X. X \rightarrow X}$, e.g.:

$$\frac{\frac{\frac{\frac{\frac{\vdash_{\text{PT}} \mu X. X \rightarrow X : \omega \rightarrow \omega}{\vdash_{\text{PT}} \mu X. X \rightarrow X : \omega \rightarrow \omega} \quad \frac{X : \omega \rightarrow \omega \vdash_{\text{PT}} X : \omega \rightarrow \omega \quad \frac{X : \omega \rightarrow \omega \vdash_{\text{PT}} \diamond \quad X \vdash_{\text{T}} X}{X : \omega \rightarrow \omega \vdash_{\text{PT}} X : \omega} (\omega)}{X : \omega \rightarrow \omega \vdash_{\text{PT}} X \rightarrow X : (\omega \rightarrow \omega) \rightarrow \omega} \text{(Type Arrow)}}{X : \omega \rightarrow \omega \vdash_{\text{PT}} X \rightarrow X : (\omega \rightarrow \omega) \rightarrow \omega} \text{(Type Rec)}}{\vdash_{\text{PT}} \mu X. X \rightarrow X : (\omega \rightarrow \omega) \rightarrow \omega}$$

■

We can naturally link derivations in \vdash_{PT} to those in \vdash_{T} via erasure of predicates.

Lemma 2.8 ERASING. *If $\Delta \vdash_{\text{PT}} \diamond$, then $\widehat{\Delta} \vdash_{\text{T}} \diamond$. Similarly, if $\Delta \vdash_{\text{PT}} A : \sigma$, then $\widehat{\Delta} \vdash_{\text{T}} A$.*

Proof: In both cases the proof is an easy induction on the structure of derivations. All cases are trivial, except for when the derivation ends by rule (\wedge) or (Type Rec) ; then the result follows by induction. ■

Lemma 2.9 *i) If $\Delta, X : \tau \vdash_{\text{PT}} B : \sigma$, then $X \notin \text{dom}(\Delta)$.*

ii) If $\Delta_0, \Delta_1 \vdash_{\text{PT}} B : \sigma$, such that no free type variable in B is declared in Δ_1 , then $\Delta_0 \vdash_{\text{PT}} B : \sigma$,

Proof: Easy. ■

The next lemma states some standard properties of \vdash_{PT} , that follow immediately from the rules in Figure 2.

Lemma 2.10 TYPE PREDICATE GENERATION LEMMA. *Let $\mathcal{D} :: \Delta \vdash_{\text{PT}} A : \sigma$. If $A = \text{Top}$, then $\sigma = \omega$; otherwise, either $\sigma = \omega$, or:*

($\sigma = \bigwedge_{i \in I} \phi_i$) : Then, for all $i \in I$ there exist $\mathcal{D}_i :: \Delta \vdash_{\text{PT}} A : \phi_i$, sub-derivations of \mathcal{D} ;

($\omega \neq \sigma \in \mathcal{P}_S$) : a) If $\mathcal{D} :: \Delta \vdash_{\text{PT}} K : \sigma$, then σ is atomic;

b) if $\mathcal{D} :: \Delta \vdash_{\text{PT}} X : \sigma$ then $X : \tau \in \Delta$ and $\tau \leq \sigma$;

c) if $\mathcal{D} :: \Delta \vdash_{\text{PT}} A : \sigma$ and $X \in \text{fv}(A)$ then $X : \tau \in \Delta$, for some τ ;

d) if $\mathcal{D} :: \Delta \vdash_{\text{PT}} A \rightarrow B : \sigma$ then there exist τ, ϕ such that $\sigma = \tau \rightarrow \phi$ and $\mathcal{D}_1 :: \Delta \vdash_{\text{PT}} A : \tau$ and $\mathcal{D}_2 :: \Delta \vdash_{\text{PT}} B : \phi$, sub-derivations of \mathcal{D} ;

- e) if $\mathcal{D} :: \Delta \vdash_{\text{PT}} [\ell_i : B_i^{(i \in I)}] : \sigma$ then for some $j \in I$, $\sigma = \langle \ell_j : \tau \rightarrow \psi \rangle$ and there exists both $\mathcal{D}' :: \Delta \vdash_{\text{PT}} [\ell_i : B_i^{(i \in I)}] : \tau$ and $\mathcal{D}'' :: \Delta \vdash_{\text{PT}} B_j : \psi$, sub-derivations of \mathcal{D} ;
- f) if $\mathcal{D} :: \Delta \vdash_{\text{PT}} \mu X.A : \sigma$, then there exists τ and $\mathcal{D}' :: \Delta \vdash_{\text{PT}} \mu X.A : \tau$ and $\mathcal{D}'' :: \Delta, X : \tau \vdash_{\text{PT}} A : \sigma$, sub-derivations of \mathcal{D} .

Proof: By straightforward induction on the structure of derivations. ■

Although the above property is defined using the structure of types, we can generalise the above to, for example, the following property, which we can show is a consequence of the previous lemma:

Lemma 2.11 If $\mathcal{D} :: \Delta \vdash_{\text{PT}} [\ell_i : B_i^{(i \in I)}] : \sigma$ and $\sigma \neq \omega$ then for some $J \subseteq I$, $\sigma = \bigwedge_{j \in J} \sigma_j$ and for each j there exists H_j such that $\sigma_j = \langle \ell_j : \tau_h \rightarrow \psi_h^{(h \in H_j)} \rangle$ and $\mathcal{D}_h :: \Delta \vdash_{\text{PT}} [\ell_i : B_i^{(i \in I)}] : \tau_h$ and $\mathcal{D}'_h :: \Delta \vdash_{\text{PT}} B_j : \psi_h$ for all $h \in H_j$; moreover all these are sub-derivations of \mathcal{D} ;

and similarly for the other syntactic constructs.

The next lemma shows that, for object types A and C , if σ is a predicate we can assign to A , which is a super-type of C , then also σ can also be assigned to C .

Lemma 2.12 Let $A \equiv [\ell_i : B_i^{(i \in I)}]$, $C \equiv [\ell_j : B_j^{(j \in J)}]$ for some $J \supseteq I$ and suppose $\widehat{\Delta} \vdash_{\tau} C <: A$. Then, for any σ , if $\Delta \vdash_{\text{PT}} A : \sigma$, then $\Delta \vdash_{\text{PT}} C : \sigma$.

Proof: If $\sigma = \omega$, the thesis is trivial by rule (ω) . For $\sigma \neq \omega$, we reason by induction on the structure derivations. Now, $\widehat{\Delta} \vdash_{\tau} C <: A$ implies that both C and A are well formed types under $\widehat{\Delta}$.

By Lemma 2.10(e) we know that σ is (equivalent to) an intersection of record types of the shape $\langle \ell_i : \tau \rightarrow \phi \rangle$ for some $i \in I \subseteq J$, and that both $\Delta \vdash_{\text{PT}} A : \tau$, for some τ , and $\Delta \vdash_{\text{PT}} B_i : \phi$ are derivable in sub-derivations. Therefore, by induction, we know that $\Delta \vdash_{\text{PT}} C : \tau$. Reconstructing the derivation as the one for $\Delta \vdash_{\text{PT}} A : \sigma$, we obtain $\Delta \vdash_{\text{PT}} C : \sigma$. ■

Lemma 2.13 If both $\Delta_0, \Delta_1 \vdash_{\text{PT}} A : \sigma$ and $\widehat{\Delta}_0 \vdash_{\tau} A$, then $\Delta_0 \vdash_{\text{PT}} A : \sigma$.

Proof: By induction on the structure of derivations. Since derivations in the Predicates to Types Assignment System mirror derivations in the type and subtype system of the object calculus in almost all cases, we focus on those rules which are not just a decoration of the type formation rules.

(ω) : Obvious, since we can use $\widehat{\Delta}_0 \vdash_{\tau} A$ as the premise of the same rule.

(\wedge) : By induction.

(Type Top) : Trivial, as the context Δ does not play any role in this case.

(Type Object) : Then $A \equiv [\ell_i : B_i^{(i \in I)}]$, and $\sigma \equiv \langle \ell_j : \tau \rightarrow \phi \rangle$ for some $j \in I$, and both $\Delta_0, \Delta_1 \vdash_{\text{PT}} A : \tau$ and $\Delta_0, \Delta_1 \vdash_{\text{PT}} B_j : \phi$. By Lemma 1.8(x), we have $\Delta_0 \vdash_{\text{PT}} B_i$ for all $i \in I$; by induction we have also $\Delta_0 \vdash_{\text{PT}} A : \tau$ and $\Delta_0 \vdash_{\text{PT}} B_j : \phi$, and we get $\Delta_0 \vdash_{\text{PT}} A : \langle \ell_j : \tau \rightarrow \phi \rangle$ by rule (Type Object).

(Type Arrow) : Then $A \equiv B \rightarrow C$, $\sigma \equiv \tau \rightarrow \phi$ and both $\Delta_0, \Delta_1 \vdash_{\text{PT}} B : \tau$ and $\Delta_0, \Delta_1 \vdash_{\text{PT}} C : \phi$. By induction both $\Delta_0 \vdash_{\text{PT}} B : \tau$ and $\Delta_0 \vdash_{\text{PT}} C : \phi$, hence $\Delta_0 \vdash_{\text{PT}} B \rightarrow C : \tau \rightarrow \phi$ by rule (Type Arrow).

(Type Rec) : Then $A \equiv \mu X.B$ and both $\Delta_0, \Delta_1 \vdash_{\text{PT}} \mu X.B : \tau$ and $\Delta, X : \tau \vdash_{\text{PT}} B : \sigma$. By induction $\Delta_0 \vdash_{\text{PT}} \mu X.B : \tau$; by Lemma 1.8(ix), $\widehat{\Delta}_0 \vdash_{\tau} \mu X.B$ implies $\widehat{\Delta}_0, X \vdash_{\tau} B$. Since no free type variable in B is declared in Δ_1 , by Lemma 2.9 we obtain $\Delta_0, X : \tau \vdash_{\text{PT}} B : \sigma$. Using the assumption $\Delta_0 \vdash_{\text{PT}} \mu X.B : \tau$, we get the desired $\Delta_0 \vdash_{\text{PT}} \mu X.B : \sigma$ by rule (Type Rec). ■

Lemma 2.14 The following rules are admissible:

(Relevance) :

$$\frac{\Delta, X : \tau, \Delta' \vdash_{\text{PT}} A : \sigma \quad \Delta, \Delta' \vdash_{\text{PT}} \diamond}{\Delta, \Delta' \vdash_{\text{PT}} A : \sigma} (X \notin \text{fv}(A))$$

(Weak) :

$$\frac{\Delta \vdash_{\text{PT}} A : \sigma}{\Delta, X : \tau \vdash_{\text{PT}} A : \sigma} (X \notin \text{dom}(\Delta))$$

$$\begin{array}{c}
(\text{Cut}) : \\
\frac{\Delta, X:\sigma \vdash_{\text{PT}} B:\tau \quad \Delta \vdash_{\text{PT}} A:\sigma}{\Delta \vdash_{\text{PT}} B\{X \leftarrow A\}:\tau}
\end{array}
\qquad
\begin{array}{c}
(\text{Cut}<:) : \\
\frac{\Delta \vdash_{\text{PT}} A:\sigma \quad \Delta, X:\sigma <: C \vdash_{\text{PT}} B:\tau \quad \widehat{\Delta} \vdash_{\text{T}} A <: C}{\Delta \vdash_{\text{PT}} B\{X \leftarrow A\}:\tau}
\end{array}$$

(\leq):

$$\frac{\Delta \vdash_{\text{PT}} A:\sigma \quad \Delta' \vdash_{\text{PT}} \diamond}{\Delta' \vdash_{\text{PT}} A:\tau} \quad (\sigma \leq \tau, \Delta' \leq \Delta)$$

Proof: By easy induction on the structure of derivations. The proof of admissibility of (Relevance) and (Weak) parallels the arguments in Lemma 1.10. In the cases of (Cut) and (Cut<:) we use Lemma 4.6. For rule (\leq), the proof is much as that for Lemma 3.9. ■

Lemma 2.15 For any pre-context Δ , pre-type $\mu X.A$, and predicate σ :

$$\Delta \vdash_{\text{PT}} \mu X.A:\sigma \iff \Delta \vdash_{\text{PT}} A\{X \leftarrow \mu X.A\}:\sigma.$$

Proof: (\Leftarrow): By induction on the shape of A .

($A \equiv X$): then $A\{X \leftarrow \mu X.A\} \equiv \mu X.A$ and the thesis is trivial.

($A \equiv K$ or $A \equiv Y \neq X$): (and in general if $X \notin \text{fv}(A)$) then $A\{X \leftarrow \mu X.A\} \equiv A$. Since $\Delta \vdash_{\text{PT}} A:\sigma$ implies $\Delta, X:\omega \vdash_{\text{PT}} A:\sigma$ by rule (Weak); we have also $\Delta \vdash_{\text{PT}} \mu X.A:\omega$ by rule (ω), we derive $\Delta \vdash_{\text{PT}} \mu X.A:\sigma$ by rule (Type Rec).

($A \equiv B \rightarrow C$): then $A\{X \leftarrow \mu X.A\} \equiv B\{X \leftarrow \mu X.A\} \rightarrow C\{X \leftarrow \mu X.A\}$: the thesis follows by induction, according to the possible cases for σ (see Lemma 2.10(d)).

($A \equiv [\ell_i; B_i \ (i \in I)]$): similar.

(\Rightarrow): for $\sigma = \omega$, the thesis is trivial; if σ is an intersection, the thesis follows by induction; else, by Lemma 2.10(f), there exists τ such that $\Delta \vdash_{\text{PT}} \mu X.A:\tau$ and $\Delta, X:\tau \vdash_{\text{PT}} A:\sigma$. The thesis now follows from rule (Cut). ■

We will now show that \vdash_{PT} is downwards closed for $<:$.

Theorem 2.16 The following rule is admissible:

$$\begin{array}{c}
(:>) : \\
\frac{\Delta \vdash_{\text{PT}} B:\sigma \quad \widehat{\Delta} \vdash_{\text{T}} A <: B}{\Delta \vdash_{\text{PT}} A:\sigma}
\end{array}$$

Proof: If $\sigma = \omega$, then by Lemma 1.8(v), $\widehat{\Delta} \vdash_{\text{T}} A <: B$ implies $\widehat{\Delta} \vdash_{\text{T}} A$, and $\Delta \vdash_{\text{PT}} A:\omega$ is derivable by rule (ω). If σ is an intersection, the result follows by induction. Otherwise, for σ strict, we reason by a principal induction on the derivation of $\widehat{\Delta} \vdash_{\text{T}} A <: B$ and a secondary induction on the derivation of $\Delta \vdash_{\text{PT}} B:\sigma$.

(Sub Refl), (Sub Trans): The first case is trivial and the second one follows by induction and the transitivity of \leq .

(Sub Top): Then $B \equiv \text{Top}$ and $\sigma = \omega$ by Lemma 2.10. As above, by Lemma 1.8(v), $\widehat{\Delta} \vdash_{\text{T}} A <: B$ implies $\widehat{\Delta} \vdash_{\text{T}} A$, so $\Delta \vdash_{\text{PT}} A:\omega$ is derivable by rule (ω).

(Sub X): Then $A \equiv X$ and $\widehat{\Delta} \equiv \widehat{\Delta}', X <: B, \widehat{\Delta}''$ for some Δ', Δ'' , and $\widehat{\Delta}$ is a well-formed context. By Lemma 1.8(vi) this implies $\widehat{\Delta}' \vdash_{\text{T}} B$. Therefore, by Lemma 2.13 and the assumption, we have $\Delta' \vdash_{\text{PT}} B:\sigma$. Now, since $\sigma \leq \sigma$, by rule (Env X<:), we have $\Delta', X:\sigma <: B \vdash_{\text{T}} \diamond$, and, by rule (Type X<:), $\Delta', X:\sigma <: B \vdash_{\text{PT}} X:\sigma$, and the thesis follows by rule (Weak).

(Sub Object): This is an immediate consequence of Lemma 2.12.

(Sub Arrow): Then $A \equiv A' \rightarrow A'', B \equiv B' \rightarrow B''$ and both $\widehat{\Delta} \vdash_{\text{T}} B' <: A'$ and $\widehat{\Delta} \vdash_{\text{T}} A'' <: B''$. Then $\sigma = \rho \rightarrow \psi$, $\Delta \vdash_{\text{PT}} B':\rho$ and $\Delta \vdash_{\text{PT}} B'':\psi$. By induction $\Delta \vdash_{\text{PT}} A'':\psi$; on the other hand, since

$\Delta \vdash_{\text{PT}} A' : \omega$, so $\Delta \vdash_{\text{PT}} A' \rightarrow A'' : \omega \rightarrow \psi$. Since $\omega \rightarrow \psi \leq \rho \rightarrow \psi$, we get $\Delta \vdash_{\text{PT}} A' \rightarrow A'' : \sigma$ by rule (\leq).

(Sub Rec₁), (Sub Rec₂) : These follow by Lemma 2.14 and 2.15.

(Sub Rec₃) : Then $A \equiv \mu X.A'$, $B \equiv \mu Y.B'$. By Lemma 2.10(f), there exists σ' and sub-derivations for $\Delta \vdash_{\text{PT}} \mu Y.B' : \sigma'$ and $\Delta, Y:\sigma' \vdash_{\text{PT}} B' : \sigma$. By secondary induction, $\Delta \vdash_{\text{PT}} \mu X.A : \sigma'$. Since $X \notin \text{dom}(\Delta)$ we have $\Delta, Y:\sigma', X:\sigma', X <: Y \vdash_{\text{T}} \diamond$.

On the other hand, applying rule (Weak) to $\Delta, Y:\sigma' \vdash_{\text{PT}} B' : \sigma$ gives $\Delta, Y:\sigma', X:\sigma' \vdash_{\text{PT}} B' : \sigma$. But we know that $\widehat{\Delta}, X, Y, X <: Y \vdash_{\text{T}} A' <: B'$ is the premise of the last inference in the derivation for $\widehat{\Delta} \vdash_{\text{T}} A <: B$, so by the principal induction $\Delta, Y:\sigma', X:\sigma' \vdash_{\text{PT}} A' : \sigma$. Now $Y \notin \text{fv}(A')$ (otherwise, since $X \neq Y$, $Y \in \text{fv}(\mu X.A')$ so that we need $Y \in \text{dom}(\Delta)$ for deducing $\Delta \vdash_{\text{PT}} \mu X.A : \sigma'$ by Lemma 2.10(c), which is not the case), hence, by Lemma 2.142.14, $\Delta, X:\sigma' \vdash_{\text{PT}} A' : \sigma$. From this and $\Delta \vdash_{\text{PT}} \mu X.A : \sigma'$ we conclude $\Delta \vdash_{\text{PT}} \mu X.A : \sigma$. ■

Corollary 2.17 LANGUAGES AND SUBTYPING. *If $\vdash_{\text{T}} A <: B$ then both A and B are closed types, and $\mathcal{L}_A \supseteq \mathcal{L}_B$. In particular if $\vdash_{\text{T}} \mu X.A$ then $\mathcal{L}_{\mu X.A} = \mathcal{L}_{A\{X \leftarrow \mu X.A\}}$.*

Proof: That A and B are closed is an immediate consequence of Lemma 1.9, hence \mathcal{L}_A and \mathcal{L}_B are well defined. Then the first part of the thesis follows by Theorem 2.16. This implies the second part together with the fact that $\vdash_{\text{T}} \mu X.A =: A\{X \leftarrow \mu X.A\}$. ■

3 The Assignment System

We now come to the definition of *predicate assignment*, where we associate predicates to typed terms.

Definition 3.1 BASES AND TERM PRE-CONTEXTS. *i) A basis Γ is defined inductively by:*

$$\Gamma ::= \emptyset \mid \Gamma, x:A:\sigma$$

where x is a term variable, A is a pre-type, and σ is a predicate. The *domain* of a basis is defined by: $\text{dom}(\Gamma, x:A:\sigma) = \{x\} \cup \text{dom}(\Gamma)$, and $\widehat{\Gamma}, x:A:\sigma = \widehat{\Gamma}, x:A$.

ii) We extend \leq to contexts by: $\Gamma \leq \Gamma'$ if and only if, for every $x:A:\sigma' \in \Gamma'$ there exists $x:A:\sigma \in \Gamma$ such that $\sigma \leq \sigma'$.

iii) A *term pre-context* is a pair $\Delta; \Gamma$ such that Δ is a predicate pre-context and Γ is a basis. We also define $\text{dom}(\Delta; \Gamma) = \text{dom}(\Delta) \cup \text{dom}(\Gamma)$ and $\widehat{\Delta; \Gamma} = \widehat{\Delta}, \widehat{\Gamma}$ (where the comma represents the concatenation of the two sequences).

Definition 3.2 ASSIGNMENT SYSTEM. *An assignment judgement is a triple $a:A:\sigma$ expressing the assignment of a predicate σ to the (pre) term a of (pre) type A . An assignment sequent has the shape $\Delta; \Gamma \vdash_{\text{T}} a:A:\sigma$ where $\Delta; \Gamma$ is a term pre-context.*

The *Assignment System* to derive assignment sequents is defined in Figure 5.

We say that a term pre-context $\Delta; \Gamma$ is a *term context* if $\Delta; \Gamma \vdash_{\text{PT}} \diamond$ is derivable in the assignment system.

The judgement $a:A:\sigma$ tells at the same time that a is of type A , and that it satisfies the predicate σ : hence this implies that $A:\sigma$ (as this is witnessed by a), which is in fact ensured by the formal system. Since ω is the trivial predicate, the judgement $a:A:\omega$ is the same as $a:A$, i.e. a has type A . This is why we use \vdash_{\circ} in the definition of \vdash_{P} . However we could avoid this explicit composition of systems, at the price of doubling all rules relating predicates to term and type structure, adding an instance of each such rule with all trivial predicates at the right end of the sequents. E.g. in the case of (Val Appl) we

Figure 5 The Predicate Assignment System

Let $A \equiv [\ell_i : B_i \ (i \in I)]$:

$$\begin{array}{c}
 \text{(Env } \emptyset \text{)} : \quad \text{(Env } x \text{)} : \quad \text{(Val } x \text{)} : \\
 \frac{\Delta \vdash_{\mathbb{T}} \diamond}{\Delta; \emptyset \vdash_{\mathbb{P}} \diamond} \quad \frac{\Delta \vdash_{\mathbb{PT}} B : \sigma \quad \Delta; \Gamma \vdash_{\mathbb{P}} \diamond}{\Delta; \Gamma, x : B : \sigma \vdash_{\mathbb{P}} \diamond} \ (x \notin \text{dom}(\Gamma)) \quad \frac{\Delta; \Gamma', x : B : \sigma, \Gamma'' \vdash_{\mathbb{P}} \diamond}{\Delta; \Gamma', x : B : \sigma, \Gamma'' \vdash_{\mathbb{P}} x : B : \psi} \ (\sigma \leq \psi) \\
 \\
 \text{(<:)} : \\
 \frac{\Delta; \Gamma \vdash_{\mathbb{P}} a : B : \psi \quad \widehat{\Delta} \vdash_{\mathbb{T}} B <: C \quad \Delta \vdash_{\mathbb{PT}} C : \psi}{\Delta; \Gamma \vdash_{\mathbb{P}} a : C : \psi} \\
 \\
 \text{(Val Fun)} : \quad \text{(Val Appl)} : \\
 \frac{\Delta; \Gamma, x : A : \sigma \vdash_{\mathbb{P}} a : B : \phi}{\Delta; \Gamma \vdash_{\mathbb{P}} \lambda x^A. a : A \rightarrow B : \sigma \rightarrow \phi} \quad \frac{\Delta; \Gamma \vdash_{\mathbb{P}} a : A \rightarrow B : \sigma \rightarrow \phi \quad \Delta; \Gamma \vdash_{\mathbb{P}} b : A : \sigma}{\Delta; \Gamma \vdash_{\mathbb{P}} a(b) : B : \phi} \\
 \\
 \text{(Val Object)} : \quad \text{(\omega)} : \\
 \frac{\Delta; \Gamma, x_j : A : \sigma \vdash_{\mathbb{P}} b_j : B_j : \phi \quad \widehat{\Delta}; \widehat{\Gamma}, x_i : A \vdash_{\mathbb{O}} b_i : B_i \ (\forall i \in I \setminus j)}{\Delta; \Gamma \vdash_{\mathbb{P}} [\ell_i = \varsigma(x_i^A) b_i \ (i \in I)] : A : \langle \ell_j : \sigma \rightarrow \phi \rangle} \ (j \in I) \quad \frac{\Delta; \Gamma \vdash_{\mathbb{P}} \diamond \quad \widehat{\Delta}; \widehat{\Gamma} \vdash_{\mathbb{O}} a : B}{\Delta; \Gamma \vdash_{\mathbb{P}} a : B : \omega} \\
 \\
 \text{(Val Update}_1 \text{)} : \quad \text{(Val Select)} : \\
 \frac{\Delta; \Gamma \vdash_{\mathbb{P}} a : A : \sigma \quad \Delta; \Gamma, y : A : \tau \vdash_{\mathbb{P}} b : B_j : \phi}{\Delta; \Gamma \vdash_{\mathbb{P}} (a.\ell_j \Leftarrow \varsigma(y^A) b) : A : \langle \ell_j : \tau \rightarrow \phi \rangle} \ (\sigma \neq \omega, j \in I) \quad \frac{\Delta; \Gamma \vdash_{\mathbb{P}} a : A : \langle \ell_j : \sigma \rightarrow \phi \rangle \quad \Delta; \Gamma \vdash_{\mathbb{P}} a : A : \sigma}{\Delta; \Gamma \vdash_{\mathbb{P}} a.\ell_j : B_j : \phi} \\
 \\
 \text{(Val Update}_2 \text{)} : \quad \text{(\wedge)} : \\
 \frac{\Delta; \Gamma \vdash_{\mathbb{P}} a : A : \langle \ell_k : \phi \rangle \quad \widehat{\Delta}; \widehat{\Gamma}, y : A \vdash_{\mathbb{O}} b : B_j}{\Delta; \Gamma \vdash_{\mathbb{P}} (a.\ell_j \Leftarrow \varsigma(y^A) b) : A : \langle \ell_k : \phi \rangle} \ (j, k \in I, k \neq j) \quad \frac{\Delta; \Gamma \vdash_{\mathbb{P}} a : B : \sigma_i \ (\forall i \in I)}{\Delta; \Gamma \vdash_{\mathbb{P}} a : B : \bigwedge_{i \in I} \sigma_i}
 \end{array}$$

would have:

$$\begin{array}{c}
 \text{(Val Appl } \omega \text{)} : \\
 \frac{\Delta; \Gamma \vdash_{\mathbb{P}} a : A \rightarrow B : \omega \quad \Delta; \Gamma \vdash_{\mathbb{P}} b : A : \omega}{\Delta; \Gamma \vdash_{\mathbb{P}} a(b) : B : \omega}
 \end{array}$$

The actual formulation of rule (ω) avoids such verbose introductions of the trivial predicate, by preserving its meaning of “being a typeable term”.

In the case of (Val Object) and of the (Val Update_i) , different formulations are reported in the Introduction, which are only apparently stronger of the corresponding rules in Figure 5: take all the predicates ϕ and ψ to be ω in the discarded parts, just to ensure typeability.

The side condition $\sigma \neq \omega$ of rule (Val Update_1) is decidable by (i) of Lemma 2.3. It is immaterial w.r.t. subject reduction and subject expansion Theorems 4.7 and 4.10 respectively, in the next section. Indeed, as far as we are concerned with these properties of the system, it could be replaced by the weaker

$$\begin{array}{c}
 \text{(Val Update}'_1 \text{)} : \\
 \frac{\widehat{\Delta}; \widehat{\Gamma} \vdash_{\mathbb{O}} a : A \quad \Delta; \Gamma, y : A : \tau \vdash_{\mathbb{P}} b : B_j : \phi}{\Delta; \Gamma \vdash_{\mathbb{P}} (a.\ell_j \Leftarrow \varsigma(y^A) b) : A : \langle \ell_j : \tau \rightarrow \phi \rangle} \ (j \in I)
 \end{array}$$

This can be understood from the observation that the rule expresses that the newly derived predicate does *not* depend on any predicate for a at all; the only thing we need for subject reduction and expansion is that a is well-formed of type A .

On the other hand it is essential for Theorem 6.10 to hold, as well as for the subsequent results. These say that any (closed) term a satisfies a non trivial predicate if and only if it is convergent, namely reduces to a value. Now the predicate $\langle \ell_j; \tau \rightarrow \phi \rangle$ is non trivial, but $a.\ell_j \Leftarrow \zeta(y^A)b$ converges only if a does. It is a remarkable fact that this is a combined effect of the type system and of the predicate system: indeed for $a.\ell_j \Leftarrow \zeta(y^A)b$ to be convergent we also need that a reduces to an object term, having a label ℓ_j . The type system ensures that if a converges, then this will be the case; the assumption that $a:A:\sigma$ for $\sigma \neq \omega$ implies that a actually converges, even if σ is discarded in the conclusion. This will be formally proved in section 6.

We now formulate the link between $\vdash_{\mathbb{T}}$ and $\vdash_{\mathbb{T}}$, and $\vdash_{\mathbb{P}}$ and $\vdash_{\mathbb{O}}$, which is a conservativity result.

Lemma 3.3 ASSIGNMENT ERASING. *If $\Delta; \Gamma \vdash_{\mathbb{P}\mathbb{T}} \diamond$ and $\Delta; \Gamma \vdash_{\mathbb{P}} a:A:\sigma$, then $\widehat{\Delta}; \widehat{\Gamma} \vdash_{\mathbb{T}} \diamond$ and $\widehat{\Delta}; \widehat{\Gamma} \vdash_{\mathbb{O}} a:A$.*

Proof: By induction on the structure of derivations using Lemma 2.8. ■

The following lemma links $\vdash_{\mathbb{P}}$ and $\vdash_{\mathbb{P}\mathbb{T}}$.

Lemma 3.4 *If $\Delta; \Gamma, x:A:\sigma \vdash_{\mathbb{P}} b:B:\tau$ then both $\Delta \vdash_{\mathbb{P}\mathbb{T}} A:\sigma$ and $\Delta \vdash_{\mathbb{P}\mathbb{T}} B:\tau$.*

Proof: By straightforward induction on derivations. It is a consequence of the fact that the assignment $\Delta \vdash_{\mathbb{O}} A:\sigma$ of a predicate to a type under the assumptions in Δ for each judgement $a:A:\sigma$ occurring in a derivation is checked by the appropriate rule. ■

We will now show that also $\vdash_{\mathbb{P}}$ is downwards closed for $<:$.

Theorem 3.5 *The following rule is admissible:*

$$\begin{array}{c} (:>) : \\ \frac{\Delta; \Gamma \vdash_{\mathbb{P}} a:C:\sigma \quad \widehat{\Delta}; \widehat{\Gamma} \vdash_{\mathbb{O}} a:A \quad \widehat{\Delta} \vdash_{\mathbb{T}} A <: C}{\Delta; \Gamma \vdash_{\mathbb{P}} a:A:\sigma} \end{array}$$

Proof: By induction on the structure of derivations in $\vdash_{\mathbb{P}}$, using Theorem 2.16. E.g. if the derivation ends by rule (Val Fun):

$$\frac{\Delta; \Gamma, x:C':\tau' \vdash_{\mathbb{P}} b:D:\psi}{\Delta; \Gamma \vdash_{\mathbb{P}} \lambda x^{C'}.b:C' \rightarrow D:\tau' \rightarrow \psi}$$

where $\widehat{\Delta} \vdash_{\mathbb{T}} A <: C \equiv C' \rightarrow D$. By Lemma ??(iii), $A = A' \rightarrow A''$, $E \vdash_{\mathbb{T}} C' <: A'$ and $E \vdash_{\mathbb{T}} A'' <: D$. By induction $\Delta; \Gamma, x:C':\tau' \vdash_{\mathbb{P}} b:A'':\psi$ hence $\Delta; \Gamma \vdash_{\mathbb{P}} \lambda x^{C'}.b:A' \rightarrow A'':\tau' \rightarrow \psi$. Take any σ' such that $\Delta \vdash_{\mathbb{T}} A':\sigma'$ and $\tau' \leq \sigma'$ (which exists, and at worst is ω), then we get $\Delta; \Gamma \vdash_{\mathbb{P}} \lambda x^{C'}.b:A' \rightarrow A'':\sigma' \rightarrow \psi$. Since $\sigma' \rightarrow \psi \leq \tau' \rightarrow \psi$, we conclude by (\leq) . ■

Lemma 3.6 *The following rule is admissible:*

$$\frac{\Delta; \Gamma, x:A:\tau, \Gamma' \vdash_{\mathbb{P}} a:C:\sigma \quad \Delta \vdash_{\mathbb{O}} A':\tau' \quad \widehat{\Delta} \vdash_{\mathbb{T}} A' <: A}{\Delta; \Gamma, x:A':\tau', \Gamma' \vdash_{\mathbb{P}} a:C:\sigma} (\tau' \leq \tau)$$

Proof: Easy induction on the structure of derivations. Informally, each time there is an instance of the rule (Env x) with conclusion $\Delta; \Gamma, x:A:\tau, \Gamma'' \vdash_{\mathbb{P}} x:A:\tau$ in the derivation for $\Delta; \Gamma, x:A:\tau, \Gamma' \vdash_{\mathbb{P}} a:C:\sigma$, we replace it by an instance of $(<:)$ whose premises are $\Delta; \Gamma, x:A':\tau', \Gamma'' \vdash_{\mathbb{P}} x:A:\tau'$, $\widehat{\Delta} \vdash_{\mathbb{T}} A' <: A$ and $\Delta \vdash_{\mathbb{O}} A':\tau'$, where the conclusion is $\Delta; \Gamma, x:A:\tau', \Gamma'' \vdash_{\mathbb{P}} x:A:\tau$. ■

The essential properties of the predicate assignment system, on which the subsequent treatment relies, are stated in next lemma.

- Lemma 3.7 PREDICATE GENERATION LEMMA.** *If $\mathcal{D} :: \Delta; \Gamma \vdash_P a:A:\sigma$, then either: $\sigma = \omega$, or, $(\sigma = \bigwedge_{i \in I} \phi_i)$: Then, for all $1 \leq i \leq n$ there exist $\mathcal{D}_i :: \Delta \vdash_P a:A:\phi_i$, sub-derivations of \mathcal{D} : $(\sigma \equiv \phi \in \mathcal{P}_S \setminus \{\omega\})$: a) if $\Delta; \Gamma \vdash_P c:A:\phi$ then c is a constant of some ground type K , with $A \equiv K$, $\phi \equiv \kappa$ and $\emptyset \vdash_{PT} K:\kappa$;*
- b) *if $\Delta; \Gamma \vdash_P x:A:\phi$ then $\Delta \vdash_{PT} A:\phi$ and there exists $x:C:\sigma \in \Gamma$ such that $\widehat{\Delta} \vdash_{\tau} C <: A$ and $\sigma \leq \phi$;*
 - c) *if $\Delta; \Gamma \vdash_P [\ell_i = \varsigma(x_i^{A_i})b_i]^{(i \in I)}:A:\phi$ then there exists $C \equiv [\ell_i:B_i]^{(i \in I)}$ for certain B_i , such that $A_i \equiv C$ for all $i \in I$, and there exists $J \subseteq I$ such that $A \equiv [\ell_j:B_j]^{(j \in J)}$; moreover $\phi \equiv \langle \ell_k:\tau \rightarrow \psi \rangle$ where $k \in J$, $\Delta; \Gamma, x_k:C:\tau \vdash_P b_k:B_k:\psi$, while for all $i \in I \setminus \{k\}$, $\widehat{\Delta}; \overline{\Gamma}, x_i:C \vdash_{\circ} b_i:B_i$;*
 - d) *if $\Delta; \Gamma \vdash_P [\ell_i = \varsigma(x_i^{A_i})b_i]^{(i \in I)}:A:\sigma$ then there exist $C \equiv [\ell_i:B_i]^{(i \in I)}$ for some B_i , such that $\widehat{\Delta} \vdash_{\tau} C <: A$, and, for all $i \in I$, $C \equiv A_i$ and $\widehat{\Delta}; \overline{\Gamma}, x_i:C \vdash_{\circ} b_i:B_i$. Also, $\sigma \equiv \langle \ell_j:\tau_j \rightarrow \psi_j \rangle$ for some $j \in I$, τ_j, ψ_j and $\Delta; \Gamma, x_j:C:\tau_j \vdash_P b_j:B_j:\psi_j$;*
 - e) *if $\Delta; \Gamma \vdash_P a.l:A:\phi$ then there exists $C \equiv [\dots, l:B, \dots]$ such that $\widehat{\Delta} \vdash_{\tau} B <: A$, and τ such that $\Delta; \Gamma \vdash_P a:C:\langle \ell:\tau \rightarrow \phi \rangle$ and $\Delta; \Gamma \vdash_P a:C:\tau$;*
 - f) *if $\Delta; \Gamma \vdash_P (a.l_j \leftarrow \varsigma(x^C)b):A:\sigma$ then $\widehat{\Delta}; \overline{\Gamma} \vdash_{\tau} C <: A$ and $C \equiv [\ell_i:B_i]^{(i \in I)}$ for certain B_i , with $j \in I$ and either:*
 - * *there are $\tau, \psi, \rho \neq \omega$ such that $\phi \equiv \langle \ell_j:\tau \rightarrow \psi \rangle$, $\Delta; \Gamma \vdash_P a:C:\rho$, and $\Delta; \Gamma, x:C:\tau \vdash_P b:B_j:\psi$;*
 - or
 - * *there exist ψ and $k \neq j$ such that $\phi \equiv \langle \ell_k:\psi \rangle$, $\Delta; \Gamma \vdash_P a:C:\phi$, and $\widehat{\Delta}; \overline{\Gamma}, x:C \vdash_{\circ} b:B_j$;*
 - g) *if $\Delta; \Gamma \vdash_P \lambda x^A.a:C:\phi$, then there exists B such that $\widehat{\Delta} \vdash_{\tau} A \rightarrow B <: C$, and $\phi \equiv \tau \rightarrow \psi$, and $\Delta; \Gamma, x:A:\tau \vdash_P a:B:\psi$;*
 - h) *if $\Delta; \Gamma \vdash_P a(b):C:\phi$, then there exist A, B such that $\widehat{\Delta} \vdash_{\tau} B <: C$, and there is τ such that both $\Delta; \Gamma \vdash_P a:A \rightarrow B:\tau \rightarrow \phi$ and $\Delta; \Gamma \vdash_P b:A:\tau$.*

Proof: By induction on the structure of derivations. We observe that in all clauses we use \equiv instead of $=$: (among types) and $=$ (among predicates). This is possible since these are all existential statements of derivability, and do not necessarily refer to sub-derivations of the given one: hence we can choose types and predicates of the right form as we need. In particular in clause (d) we have $\phi \equiv \langle \ell_k:\tau \rightarrow \psi \rangle$ instead of $\phi \geq \langle \ell_k:\tau \rightarrow \psi \rangle$ as one might expect. This is a consequence of the fact that languages are upward closed w.r.t. \leq (by lemma 2.14), and that $E \vdash_{\tau} A <: A$ for any A , so that if $\Delta; \Gamma \vdash_P a:A:\sigma$ and $\sigma \leq \tau$ then $\Delta; \Gamma \vdash_P a:A:\tau$ by ($<:$). Similar remarks apply to all other clauses. ■

Remark 3.8 The last lemma, together with Lemmas 1.9 and 2.10 forms the basic tool for reconstructing types and predicates in terms of the structure of the subject, namely the term. In particular Lemma 2.10 applies the same technique to the “subject” A in the judgement $A:\sigma$.

Their proofs are just backward readings of the derivation rules, and as such are done by very simple induction on derivations which we omit.

All the implications in these lemmas are actually equivalences: indeed the opposite implications follow by direct application of (possibly more than one) derivation rules.

Although the relation \leq is only used for variables, we can show the following lemma.

Lemma 3.9 *The rule*

$$(\leq) : \frac{\Delta; \Gamma \vdash_P a:A:\sigma \quad \Delta'; \Gamma' \vdash_{\tau} \diamond}{\Delta'; \Gamma' \vdash_P a:A:\tau} (\sigma \leq \tau, \Delta' \leq \Delta, \Gamma' \leq \Gamma)$$

is admissible.

Proof: ($\tau \in \mathcal{P}_s$) : By induction on the structure of the derivation for $\Delta; \Gamma \vdash_P a:A : \sigma$.

(Val x) : Then $a = x$, $\sigma \in \mathcal{P}_s$, and there exists τ such that $\Gamma = \Gamma_1, x:A:\rho, \Gamma_2$, $\rho \leq \sigma$ and $\Delta; \Gamma \vdash_{PT} \diamond$. Since $\Gamma' \leq \Gamma$, there exists $x:A:\rho' \in \Gamma'$ such that $\rho' \leq \rho$. Notice that then $\rho' \leq \tau$, and by Lemma 2.14 also $\Delta'; \Gamma' \vdash_{PT} \diamond$. Then, by rule (Val x), $\Delta', \Gamma' \vdash_P x:A : \tau$.

(<:) : Then $\Delta; \Gamma \vdash_P a:\widehat{B} : \sigma$, $\widehat{\Delta} \vdash_{PT} B <: A$, and $\widehat{\Delta} \vdash_{PT} A : \sigma$, for some B . We get $\Delta'; \Gamma' \vdash_P a:B : \tau]$ by induction; since $\widehat{\Delta'} \subseteq \widehat{\Delta}$, by weakening $\widehat{\Delta'} \vdash_{PT} B <: A$; and by Lemma 2.14 $\Delta' \vdash_{PT} A : \tau$. Then, by rule (<:) we get $\Delta'; \Gamma' \vdash_P a:A : \tau$.

(Val Fun) : Then $\sigma = \rho \rightarrow \phi$, $a = \lambda x^A. a'$, and $\Delta; \Gamma, x:A:\rho \vdash_P a':B : \phi$ for some B . Since $\sigma \leq \tau \in \mathcal{P}_s$, $\tau = \rho' \rightarrow \phi'$ with $\rho' \leq \rho$, $\phi \leq \phi'$, so, by induction, $\Delta'; \Gamma', x:A:\rho' \vdash_P a':B : \phi'$. Then, by rule (Val Fun), $\Delta'; \Gamma' \vdash_P \lambda x^A. a':A \rightarrow B : \rho' \rightarrow \phi'$.

(Val Appl) : Then $a = a_1 a_2$, and there exists ρ such that $\Delta; \Gamma \vdash_P a_1:A \rightarrow B : \rho \rightarrow \sigma$ and $\Delta; \Gamma \vdash_P a_2:A : \rho$. Since $\rho \rightarrow \sigma \leq \rho \rightarrow \tau$, by induction, $\Delta'; \Gamma' \vdash_P a_1:A \rightarrow B : \rho \rightarrow \tau$ and $\Delta'; \Gamma' \vdash_P a_2:A : \rho$. Then, by rule (Val Appl), also $\Delta'; \Gamma' \vdash_P a(b):B : \tau$.

(Val Object), (Val Update₁) : As for rule (Val Fun).

(Val Select) : As for rule (Val Appl).

(Val Update₂), ($\wedge I$) : By induction; for the latter, use Lemma 2.4.

(ω) : By assumption, $\Delta'; \Gamma' \vdash_{PT} \diamond$; since $\widehat{\Delta'}, \Gamma' \subseteq \widehat{\Delta}, \Gamma$, by weakening also $\widehat{\Delta'}, \Gamma' \vdash_{\circ} a : A$. Then, by rule (ω), $\Delta'; \Gamma' \vdash_P a:A : \omega$.

($\tau = \bigwedge_{j \in J} \tau_j$) : Assume, without loss of generality, that $\sigma = \bigwedge_{i \in I} \sigma_i$; then, by Lemma 2.4, for every $j \in J$ there is an $i \in I$ such that $\sigma_i \leq \tau_j \in \mathcal{P}_s$. The result follows by the first part of the proof, and applying rule ($\wedge I$). ■

4 Subject Reduction and Expansion

In this section we will show that predicate assignment as defined above is not only preserved by reduction, but also by expansion, i.e. if $\Delta; \Gamma \vdash_P a:A : \sigma$ and we can relate a to a' via the reduction system, then also $\Delta; \Gamma \vdash_P a':A : \sigma$.

Definition 4.1 REDUCTION. *i) Evaluating contexts* are term expressions with a hole $[-]$, and are generated by the grammar:

$$\mathcal{E}[-] ::= _ \mid \mathcal{E}[-].\ell \mid \mathcal{E}[-].\ell \Leftarrow \varsigma(x^A)b \mid \mathcal{E}[-](a).$$

We will write $\mathcal{E}[a]$ for the replacement of $_$ by a in \mathcal{E} .

ii) The one-step reduction relation on terms is the binary relation defined by the following rules:

$$\begin{aligned} [\ell_i = \varsigma(x_i^{A_i})b_i \ (i \in I)].\ell_j &\longrightarrow b_j \{x_j \leftarrow [\ell_i = \varsigma(x_i^{A_i})b_i \ (i \in I)]\} \\ [\ell_i = \varsigma(x_i^{A_i})b_i \ (i \in I)].\ell_j \Leftarrow \varsigma(x^A)b &\longrightarrow [\ell_i = \varsigma(x_i^{A_i})b_i \ (i \in I \setminus j), \ell_j = \varsigma(x^A)b] \\ (\lambda x^A. a)(b) &\longrightarrow a \{x \leftarrow b\} \\ a &\longrightarrow b \Rightarrow \mathcal{E}[a] \longrightarrow \mathcal{E}[b] \end{aligned}$$

iii) The relation \longrightarrow^ is the reflexive and transitive closure of \longrightarrow .*

The reduction relation is essentially the same in [17]. It is trivially confluent. Even relaxing Definition 4.1 and taking the closure of \longrightarrow under arbitrary contexts would not destroy confluence, as it can be shown e.g. by adapting the Martin-Löf technique for proving the Church-Rosser theorem for the λ -calculus. As for typed λ -calculi with recursion (e.g. PCF), typed terms do not necessarily have a normal form: $\Omega_B \equiv [\ell = \varsigma(x^A)x.\ell].\ell$ is typeable by B if A is any object type $[\ell:B, \dots]$, and it is such that $\Omega_B \longrightarrow \Omega_B$.

In [1], Ch. 6 the operational semantics of the object calculi is defined by means of a big-step predicate $a \rightsquigarrow v$, where a is a closed term, and v is a *value*. Values are defined as follows:

Definition 4.2 VALUES. A *value* is a closed pre-term belonging to the set defined by the grammar:

$$v ::= c \mid \lambda x^A. a \mid [\ell_i = \varsigma(x_i^A) b_i^{i \in I}].$$

It is easy to see that $a \rightsquigarrow v$ if and only if $a \xrightarrow{*} v$. The reduction relation is more general since it is defined for any term (possibly with free variable occurrences). It is even true that normal forms are not necessarily values, but if a is typeable (i.e. a term), having a normal form b , then b is value, as follows by Theorem 4.5 below and the fact that closed normal forms which are not values are not typeable.

We just stress that, consistent with the definition of \rightsquigarrow in [1], in the clause:

$$[\ell_i = \varsigma(x_i^{A_i}) b_i^{(i \in I)}]. \ell_j \Leftarrow \varsigma(x^A) b \longrightarrow [\ell_i = \varsigma(x_i^{A_i}) b_i^{i \in I \setminus j}, \ell_j = \varsigma(x^{A_j}) b]$$

a renaming of the self type of the bound variable x^A into x^{A_j} occurs. This is immaterial in the fragments of the ς -calculus without subtyping, but it is needed in the presence of rule (Val Subsumption) since if $A = [\ell_i : B_i^{i \in I}]$, and $A <: C$, then we can give type C to any term of type A and therefore update a method in an object of type A with $\varsigma(x^C) b$; but the result of (naively) performing the update saving the self type C is no longer typeable, as the *selves* of the methods now have different types, so that rule (Val Object) will not apply.

The simply typed λ -calculus is a sub-calculus of the Object Calculus: since the only evaluation contexts dealing with abstraction and application have the shape $\mathcal{E}[-](a)$, it is a lazy λ -calculus, in the sense of [5]. Also objects are “lazy”, in the sense that the bodies of the methods are not reducible before selection.

The following lemmas concerns properties of the Object Calculus whose proofs are straightforward inductions on derivations.

Lemma 4.3 *i) Let $\Delta; \Gamma \vdash_{\mathbb{P}} a : A : \sigma$, and $\Gamma' = \{x : A : \tau \in \Gamma \mid x \in \text{fv}(a)\}$, then $\Delta; \Gamma' \vdash_{\mathbb{P}} a : A : \sigma$,
ii) if $\Delta; \Gamma \vdash_{\mathbb{P}} a : A : \sigma$, and $x \in \text{fv}(a)$, then $x \in \text{dom}(\Gamma)$.*

Proof: Easy. ■

Lemma 4.4 SUBSTITUTION LEMMA FOR \vdash_0 . *If $E, x : A \vdash_0 b : B$ and $E \vdash_0 a : A$ then $E \vdash_0 b\{x \leftarrow a\} : B$.*

Proof: By induction on the structure of derivations using Lemma 1.9. The proof is similar to that of Lemma 4.6. ■

Using Lemma 4.4, we can prove the following theorem:

Theorem 4.5 SUBJECT REDUCTION FOR \vdash_0 . *If $E \vdash_0 a : A$ and $a \longrightarrow b$, then $E \vdash_0 b : A$.*

Proof: The proof is similar to that of Theorem 4.7 and simpler, by induction on the definition of \longrightarrow and using Lemma 1.9. ■

In the remaining part of this section we show that predicate assignment is closed for reduction and expansion. First we establish a substitution lemma.

Lemma 4.6 SUBSTITUTION LEMMA FOR $\vdash_{\mathbb{P}}$. *If $\Delta; \Gamma, x : A : \sigma \vdash_{\mathbb{P}} b : B : \tau$ and $\Delta; \Gamma \vdash_{\mathbb{P}} a : A : \sigma$, then $\Delta; \Gamma \vdash_{\mathbb{P}} b\{x \leftarrow a\} : B : \tau$.*

Proof: By straightforward induction on the structure of derivations, of which we show only the interesting cases; the others follow by easy induction.

(Val x) : Then either:

$(b = x)$: Then $\sigma \leq \tau$. Since $x\{x \leftarrow a\} = a$, the result follows from the second assumption and Lemma 3.9.

$(b = y \neq x)$: Since $y\{x \leftarrow a\} = y$, and $\Delta; \Gamma, x:A:\sigma \vdash_{\mathbb{P}} y:B:\tau$, by Lemma 4.3(i) we obtain $\Delta; \Gamma \vdash_{\mathbb{P}} y:B:\tau$.

(ω) : Then $\widehat{\Delta}; \widehat{\Gamma}, x:A \vdash_{\mathbb{O}} b : B$. Since by Lemma 4.4, $\widehat{\Delta}; \widehat{\Gamma} \vdash_{\mathbb{O}} b\{x \leftarrow a\} : B$, we can apply rule (ω) to get $\Delta; \Gamma \vdash_{\mathbb{P}} b\{x \leftarrow a\} : B : \omega$.

$(\wedge I)$: Then $\tau = \bigwedge_{i \in I} \tau_i$, and, for $i \in I$, $\Delta; \Gamma, x:A:\sigma \vdash_{\mathbb{P}} b:B:\tau_i$. Then $\Delta; \Gamma \vdash_{\mathbb{P}} b\{x \leftarrow a\} : B : \tau_i$ follows by induction, and, by rule $(\wedge I)$, $\Delta; \Gamma \vdash_{\mathbb{P}} b\{x \leftarrow a\} : B : \bigwedge_{i \in I} \tau_i$. \blacksquare

We use this lemma to show the following result.

Theorem 4.7 SUBJECT REDUCTION FOR $\vdash_{\mathbb{P}}$. *If $\Delta; \Gamma \vdash_{\mathbb{P}} a:A:\sigma$, and $a \longrightarrow a'$, then $\Delta; \Gamma \vdash_{\mathbb{P}} a':A:\sigma$.*

Proof: By induction on the length of the reduction sequence, of which we only show the base case, which is by definition on the reduction relation \longrightarrow . First we deal with $\omega \neq \sigma \in \mathcal{P}_S$.

$((\lambda x^D.a)(b) \longrightarrow a\{x \leftarrow b\})$: by Lemma 3.7, there exist B, C, E, τ such that both $\widehat{\Delta} \vdash_{\mathbb{T}} B <: A$ and $\widehat{\Delta} \vdash_{\mathbb{T}} D \rightarrow E <: C \rightarrow B$, and such that both $\Delta; \Gamma, x:D:\tau \vdash_{\mathbb{P}} a:E:\phi$, and $\Delta; \Gamma \vdash_{\mathbb{P}} b:C:\tau$. Notice that $C <: D$, so also $\Delta; \Gamma \vdash_{\mathbb{P}} b:D:\tau$; the result then follows from Lemma 4.6 and rule $(<:)$.

$([\ell_i = \varsigma(x_i^{A_i})b_i \ (i \in I)].\ell_j \longrightarrow b_j\{x_j \leftarrow [\ell_i = \varsigma(x_i^{A_i})b_i \ (i \in I)]\})$: as the previous part.

$([\ell_i = \varsigma(x_i^{D_i})b_i \ (i \in I)].\ell_j \leftarrow \varsigma(x^C)b \longrightarrow [\ell_i = \varsigma(x_i^{D_i})b_i \ (i \in I \setminus j), \ell_j = \varsigma(x^C)b])$: by Lemma 3.7, $j \in I$, $\widehat{\Delta} \vdash_{\mathbb{T}} C <: A$ and $C \equiv [\ell_i : C_i \ (i \in I)]$ and either:

- there are $\tau, \psi, \rho \neq \omega$ such that $\sigma \equiv \langle \ell_j : \tau \rightarrow \psi \rangle$, and both (1) $\Delta; \Gamma \vdash_{\mathbb{P}} [\ell_i = \varsigma(x_i^{D_i})b_i \ (i \in I)] : C : \rho$, and (2) $\Delta; \Gamma, x_j:C:\tau \vdash_{\mathbb{P}} b_j:C_j:\psi$. Then from (1), by Lemma 1.9, we have that there exist $E \equiv [\ell_i : E_i \ (i \in I)]$ such that $\widehat{\Delta}; \widehat{\Gamma} \vdash_{\mathbb{T}} E <: C$, and, for all $i \in I$, $E \equiv D_i$ and $\widehat{\Delta}; \widehat{\Gamma}, x_i:E \vdash_{\mathbb{O}} b_i : E_i$. Notice that, by Lemma 3.6, since $E <: C$ we also have $\Delta; \Gamma, x_j:E:\tau_j \vdash_{\mathbb{P}} b_j:E_j:\psi_j$. Then the result follows by rules (Val Object)

$$\frac{\Delta; \Gamma, x_j:E:\tau \vdash_{\mathbb{P}} b_j:E_j:\psi \quad \widehat{\Delta}; \widehat{\Gamma}, x_i:E \vdash_{\mathbb{O}} b_i : E_i \quad (\forall i \in I \setminus j)}{\Delta; \Gamma \vdash_{\mathbb{P}} [\ell_i = \varsigma(x_i^{D_i})b_i \ (i \in I \setminus j), \ell_j = \varsigma(x^C)b] : E : \langle \ell_j : \tau \rightarrow \psi \rangle} \quad (j \in I)$$

and $(<:)$, since $E <: C <: A$.

- $\sigma \equiv \langle \ell_j : \phi \rangle$, for some $j \in I$, and we have both (1) $\Delta; \Gamma \vdash_{\mathbb{P}} [\ell_i = \varsigma(x_i^{D_i})b_i \ (i \in I)] : C : \sigma$, and (2) $\widehat{\Delta}; \widehat{\Gamma}, x:C \vdash_{\mathbb{O}} b : C_k$ for some $j \neq k \in I$. From (1), by Lemma 3.7, there exist $E \equiv [\ell_i : E_i \ (i \in I)]$ such that $\widehat{\Delta}; \widehat{\Gamma} \vdash_{\mathbb{T}} E <: C$ and $E \equiv D_i$ and $\widehat{\Delta}; \widehat{\Gamma}, x_i:E \vdash_{\mathbb{O}} b_i : E_i$ for all $i \in I \setminus j$, and there exists τ, ψ such that $\phi \equiv \tau \rightarrow \psi$, and $\Delta; \Gamma, x_k:E:\tau \vdash_{\mathbb{P}} b_k:E_k:\psi$. Also, from (2), we get by Lemma 2.16 and $(<:)$ that $\widehat{\Delta}; \widehat{\Gamma}, x:E \vdash_{\mathbb{O}} b : E_j$.

Then the result follows by rules (Val Object)

$$\frac{\Delta; \Gamma, x_k:E:\tau \vdash_{\mathbb{P}} b_k:E_k:\psi \quad \widehat{\Delta}; \widehat{\Gamma}, x_i:E \vdash_{\mathbb{O}} b_i : E_i \quad (\forall i \in I \setminus k, j) \quad \widehat{\Delta}; \widehat{\Gamma}, x:E \vdash_{\mathbb{O}} b : E_j}{\Delta; \Gamma \vdash_{\mathbb{P}} [\ell_i = \varsigma(x_i^{D_i})b_i \ (i \in I \setminus j), \ell_j = \varsigma(x^E)b] : E : \langle \ell_j : \tau \rightarrow \psi \rangle} \quad (j \in I)$$

and $(<:)$, since $E <: C <: A$.

$(a \longrightarrow b \Rightarrow \mathcal{E}[a] \longrightarrow \mathcal{E}[b])$: By induction on the structure of evaluating contexts.

For $\sigma = \omega$, the result follows from Lemma 3.3, Theorem 4.5 and rule (ω) . For $\sigma = \bigwedge_{i \in I} \sigma_i$, the result follows by the strict case, and rule $(\wedge I)$. \blacksquare

Example 4.8 To better appreciate the importance of this standard result in the present setting, let us look at the following example.

Suppose that $A \equiv [\ell_0 : Int, \ell_1 : Int]$ as in Example 2.7, where the predicates in \mathcal{L}_A to be used below have been derived. Moreover let $a \equiv [\ell_0 = \varsigma(x^A)1, \ell_1 = \varsigma(x^A)x.\ell_0]$ (using a constant 1 of type Int),

so that we have $\vdash_{\mathcal{O}} a : A$ is derivable in the Object Calculus. Then (omitting the obvious parts):

$$\frac{\frac{\frac{}{x:A:\omega \vdash_{\mathcal{P}} 1: \text{Int} : \mathcal{O}}}{x:A:\langle \ell_0:\omega \rightarrow \mathcal{O} \rangle \vdash_{\mathcal{P}} x:A:\langle \ell_0:\omega \rightarrow \mathcal{O} \rangle} \quad \frac{\frac{}{x:A:\langle \ell_0:\omega \rightarrow \mathcal{O} \rangle \vdash_{\mathcal{P}} x:A:\omega} \quad (\omega)}{x:A:\langle \ell_0:\omega \rightarrow \mathcal{O} \rangle \vdash_{\mathcal{P}} x.l_0:\text{Int} : \mathcal{O}} \quad (\text{Val Select})}{\vdash_{\mathcal{P}} a:A:\langle \ell_0:\omega \rightarrow \mathcal{O}, \ell_1:\langle \ell_0:\omega \rightarrow \mathcal{O} \rangle \rightarrow \mathcal{O} \rangle} \quad (\text{Val Object}, \wedge I)}$$

where ℓ_0 is a field, ℓ_1 is the method $\text{get}\ell_0$, and we are assuming that $\mathcal{O} \in \mathcal{L}_{\text{Int}}$ is the predicate of being an odd integer. Using rules (Val Update₁), (Val Update₂) and ($\wedge I$) one can derive (the seemingly incorrect):

$$\frac{\frac{}{\vdash_{\mathcal{T}} a:A:\langle \ell_0:\omega \rightarrow \mathcal{O}, \ell_1:\langle \ell_0:\omega \rightarrow \mathcal{O} \rangle \rightarrow \mathcal{O} \rangle} \quad \frac{}{y:A:\omega \vdash_{\mathcal{P}} 2:\text{Int} : \mathcal{E}} \quad (\text{Val Update}_i, i = 1, 2, \wedge I)}{\vdash_{\mathcal{P}} (a.l_0 \Leftarrow \varsigma(y^A)2):A:\langle \ell_0:\omega \rightarrow \mathcal{E}, \ell_1:\langle \ell_0:\omega \rightarrow \mathcal{O} \rangle \rightarrow \mathcal{O} \rangle}$$

where $\mathcal{E} \in \mathcal{L}_{\text{Int}}$ is the predicate of being an even integer. This makes sense, however, since it simply states that if the value at ℓ_0 is an odd integer, then the method ℓ_1 will return an odd integer; it also states that this is vacuously true of the actual object $a.l_0 \Leftarrow \varsigma(y^A)2$, since it has an even integer at ℓ_0 . As a consequence of Theorem 4.7 we also know that this is harmless: indeed $(a.l_0 \Leftarrow \varsigma(y^A)2).l_1 \xrightarrow{*} 2$ and we clearly assume that $\not\vdash 2:\text{Int}:\mathcal{O}$, so by contraposition $\not\vdash (a.l_0 \Leftarrow \varsigma(y^A)2).l_1:\text{Int}:\mathcal{O}$. As a matter of fact, rule (Val Select) is not applicable, since $\not\vdash (a.l_0 \Leftarrow \varsigma(y^A)2):A:\langle \ell_0:\omega \rightarrow \mathcal{O} \rangle$.

On the other hand, the following odd-looking assignment is legal as well:

$$\frac{\frac{\frac{}{x:A:\langle \ell_0:\omega \rightarrow \mathcal{E} \rangle \vdash_{\mathcal{P}} x:A:\langle \ell_0:\omega \rightarrow \mathcal{E} \rangle} \quad \frac{}{x:A:\langle \ell_0:\omega \rightarrow \mathcal{E} \rangle \vdash_{\mathcal{P}} x:A:\omega} \quad (\text{Val Select})}{x:A:\omega \vdash_{\mathcal{P}} 1:\text{Int} : \mathcal{O}} \quad \frac{}{x:A:\langle \ell_0:\omega \rightarrow \mathcal{E} \rangle \vdash_{\mathcal{P}} (x.l_0):\text{Int} : \mathcal{E}} \quad (\text{Val Object}, \wedge I)}{a \vdash_{\mathcal{O}} A:\langle \ell_0:\omega \rightarrow \mathcal{O}, \ell_1:\langle \ell_0:\omega \rightarrow \mathcal{E} \rangle \rightarrow \mathcal{E} \rangle}$$

In the last case, however, the apparently wrong predicate we deduce is of use to conclude as before:

$$\frac{\frac{}{\vdash_{\mathcal{P}} a:A:\langle \ell_0:\omega \rightarrow \mathcal{O}, \ell_1:\langle \ell_0:\omega \rightarrow \mathcal{E} \rangle \rightarrow \mathcal{E} \rangle} \quad \frac{}{y:A:\omega \vdash_{\mathcal{P}} 2:\text{Int} : \mathcal{E}} \quad (\text{Val Update}_i, i = 1, 2, \wedge I)}{(a.l_0 \Leftarrow \varsigma(y^A)2) \vdash_{\mathcal{O}} A:\langle \ell_0:\omega \rightarrow \mathcal{E}, \ell_1:\langle \ell_0:\omega \rightarrow \mathcal{E} \rangle \rightarrow \mathcal{E} \rangle}$$

which is what we expected.

We now come to the proof that predicate assignment is closed for subject expansion as well. With respect to the subject reduction property there is an asymmetry, since the expansion property does not hold, in general, in $\vdash_{\mathcal{O}}$. It might seem to be contradictory w.r.t. the conservativity established in Lemma 3.3, but it is not: remember that we assign predicates to typeable terms, while the property which we are going to establish concerns the predicates, and not the types. Therefore, we do not just assume that $a' \longrightarrow a$, rather also that both have the same type A , and show that for any predicate σ such that $a:A:\sigma$ can be derived in a suitable context, also $a':A:\sigma$ can be derived in the same context. This could be called a *typed subject expansion* property.

We need the following lemma.

Lemma 4.9 EXPANSION LEMMA FOR $\vdash_{\mathcal{P}}$. Assume $\Delta; \Gamma \vdash_{\mathcal{P}} b\{x \Leftarrow a\}:B:\tau$, and both $\widehat{\Delta}; \widehat{\Gamma}, x:A \vdash_{\mathcal{O}} b:B$ and $\widehat{\Delta}; \widehat{\Gamma} \vdash_{\mathcal{O}} a:A$ for some A . Then there exist σ such that $\Delta; \Gamma, x:A:\sigma \vdash_{\mathcal{P}} b:B:\tau$ and $\Delta; \Gamma \vdash_{\mathcal{P}} a:A:\sigma$.

Proof: By induction on the structure of terms; we only show some interesting cases. Let $B = [\ell_k:B_i \text{ }^{(k \in I)}]$, and assume $\omega \neq \tau \in \mathcal{P}_s$.

- $(b = x)$: Since $x\{x \leftarrow a\} = a$, we get $\Delta; \Gamma \vdash_P a : B : \tau$. From $\widehat{\Delta}; \widehat{\Gamma}, x : A \vdash_0 x : B$, by Lemma 3.7, we get $A <: B$. Then, from $\widehat{\Delta}; \widehat{\Gamma} \vdash_0 a : A$ and $A <: B$, by Theorem 3.5, we get $\Delta; \Gamma \vdash_P a : A : \tau$. Take $\sigma = \tau$, then also by rule ($<:$), we have $\Delta; \Gamma, x : A : \tau \vdash_P x : B : \tau$.
- $(b = y \neq x)$: Since $y\{x \leftarrow a\} = y$, we get $\Delta; \Gamma \vdash_P y : B : \tau$, and, by Lemma 4.3, $\Delta; \Gamma, x : A : \omega \vdash_P y : B : \tau$. Notice that, from the fact that $\widehat{\Delta}; \widehat{\Gamma} \vdash_0 a : A$, we get, by rule (ω), $\Delta; \Gamma \vdash_P a : A : \omega$.
- $(b = c.l \Leftarrow \varsigma(y^C)d)$: If $\Delta; \Gamma \vdash_P (c.l \Leftarrow \varsigma(y^C)d)\{x \leftarrow a\} : B : \tau$ then, by the definition of substitution, $\Delta; \Gamma \vdash_P c\{x \leftarrow a\}.l \Leftarrow \varsigma(y^C)d\{x \leftarrow a\} : B : \tau$. From $\widehat{\Delta}; \widehat{\Gamma}, x : A \vdash_0 c.l \Leftarrow \varsigma(y^C)d : B$, by Lemma 1.9 we have both $\widehat{\Delta}; \widehat{\Gamma}, x : A \vdash_0 c : B$ and $\widehat{\Delta}; \widehat{\Gamma}, x : A \vdash_0 d : D$, for some D . Also by Lemma 3.7, $\widehat{\Delta} \vdash_{\tau} C <: B$ and $C \equiv [\ell_i : C_i^{(i \in I)}]$ with $j \in I$ and either:
- $\tau \equiv \langle \ell_j : \tau' \rightarrow \psi \rangle$, and we have $\Delta; \Gamma \vdash_P c\{x \leftarrow a\} : C : \rho$ and $\Delta; \Gamma, x : C : \mu \vdash_P d\{x \leftarrow a\} : D : \psi$, for some ρ, μ, τ', ψ . By induction, $\Delta; \Gamma, x : A : \sigma_1 \vdash_P c : C : \alpha$ and $\Delta; \Gamma \vdash_P a : D : \sigma_1$ for some σ_1 .
 - $\tau \equiv \langle \ell_j : \phi \rangle$, and both $\Delta; \Gamma \vdash_P c\{x \leftarrow a\} : C : \tau$ and $\Delta; \Gamma, x : C : \mu \vdash_P d\{x \leftarrow a\} : D : \psi$ for some ϕ, ψ . By induction, there exists σ_2 such that $\Delta; \Gamma, x : A : \sigma_2 \vdash_P d : C : \alpha$ and $\Delta; \Gamma \vdash_P a : D : \sigma_2$.
- In either case, take $\sigma = \sigma_1 \wedge \sigma_2$, then, by either rule (Val Update₁) or (Val Update₂), by Lemma 4.3 we get $\Delta; \Gamma, x : A : \sigma \vdash_P (c.l \Leftarrow \varsigma(y^C)d) : B : \tau$ and by rule ($\wedge I$), $\Delta; \Gamma \vdash_P a : A : \sigma$, and the result follows from rule ($<:$).
- $(b = c(d))$: If $\Delta; \Gamma \vdash_P c\{x \leftarrow a\}(d\{x \leftarrow a\}) : B : \tau$, and, by Lemma 3.7 there exists $\rho, C, A <: B$ such that $\Delta; \Gamma \vdash_P c\{x \leftarrow a\} : C \rightarrow A : \rho \rightarrow \tau$ and $\Delta; \Gamma \vdash_P d\{x \leftarrow a\} : C : \sigma$. From the assumption $\widehat{\Delta}; \widehat{\Gamma}, x : A \vdash_0 c(d) : B$, by Lemma 1.9, $\widehat{\Delta}; \widehat{\Gamma}, x : A \vdash_0 c : C \rightarrow A$ and $\widehat{\Delta}; \widehat{\Gamma}, x : A \vdash_0 d : C$. Then, by induction, there exists σ_1, σ_2 such that $\Delta; \Gamma, x : A : \sigma_1 \vdash_P c : C \rightarrow A : \rho \rightarrow \tau$ and $\Delta; \Gamma \vdash_P a : A : \sigma_1$, and $\Delta; \Gamma, x : A : \sigma_2 \vdash_P d : C : \rho$ and $\Delta; \Gamma \vdash_P a : A : \sigma_2$. Then by Lemma 4.3 and rule (Val Appl) we get $\Delta; \Gamma, x : A : \sigma_1 \wedge \sigma_2 \vdash_P c(d) : A : \tau$ and by rule ($\wedge I$), $\Delta; \Gamma \vdash_P a : A : \sigma_1 \wedge \sigma_2$. \blacksquare

Theorem 4.10 SUBJECT EXPANSION FOR \vdash_P . *If $\Delta; \Gamma \vdash_P a : A : \tau$, and a' is such that $\widehat{\Delta}; \widehat{\Gamma} \vdash_0 a' : A$ and $a' \longrightarrow a$, then $\Delta; \Gamma \vdash_P a' : A : \tau$.*

Proof: By induction on the length of the reduction sequence, of which we only show the base case, which is by definition on the reduction relation \longrightarrow . Most cases depend straightforwardly on Lemma 4.9; we show one case, that does not. First we deal with $\omega \neq \sigma \in \mathcal{P}_s$.

(i) : $[\ell_i = \varsigma(x_i^{B_i})b_i^{(i \in I)}].\ell_j \Leftarrow \varsigma(x_j^C)b_j \longrightarrow [\ell_i = \varsigma(x_i^{B_i})b_i^{i \in I \setminus j}, \ell_j = \varsigma(y^C)b]$. If

$$\Delta; \Gamma \vdash_P [\ell_i = \varsigma(x_i^{B_i})b_i^{i \in I \setminus j}, \ell_j = \varsigma(y^C)b] : A : \tau$$

then, by Lemma 3.7, there exist $D \equiv [\ell_i : D_i^{(i \in I)}]$ such that $\widehat{\Delta} \vdash_{\tau} D <: A$, $D \equiv C$, and for all $i \in I$, $B_i \equiv D$. Then $\sigma \equiv \langle \ell_k : \tau \rightarrow \psi \rangle$ for some $\tau, \psi, k \in I$, and, for all $i \in I \setminus j$, $\Delta, \Gamma, x_i : D \vdash_0 b_i : D_i$, and $\Delta, \Gamma, y : D \vdash_0 b : D_j$. By Lemma 1.9, the assumption

$$\widehat{\Delta}; \widehat{\Gamma} \vdash_0 [\ell_i = \varsigma(x_i^D)b_i^{(i \in I)}].\ell_j \Leftarrow \varsigma(y^D)b : A$$

gives $D <: A$, and $\widehat{\Delta}; \widehat{\Gamma} \vdash_0 [\ell_i = \varsigma(x_i^D)b_i^{(i \in I)}] : D$, and for all $i \in I$, $\widehat{\Delta}; \widehat{\Gamma}, x_i : D \vdash_0 b_i : D_i$, so, in particular, $\widehat{\Delta}; \widehat{\Gamma}, x_j : D \vdash_0 b_j : D_j$.

If $j = k$ and $h \neq j$, we can now construct:

$$\frac{\Delta; \Gamma, x_h : D : \omega \vdash_P b_h : D_h : \omega \quad \widehat{\Delta}; \widehat{\Gamma}, x_i : D \vdash_0 b_i : D_i \quad (\forall i \in I \setminus \{h\})}{\Delta; \Gamma \vdash_P [\ell_i = \varsigma(x_i^{D_i})b_i^{(i \in I)}] : D : \langle \ell_h : \omega \rightarrow \omega \rangle} \text{(Val Object)}$$

from which, using rule (Val Update₁):

$$\frac{\Delta; \Gamma \vdash_P [\ell_i = \varsigma(x_i^{D_i})b_i^{(i \in I)}] : D : \langle \ell_h : \omega \rightarrow \omega \rangle \quad \Delta; \Gamma, x_j : D : \tau \vdash_P b_j : D_j : \psi}{\Delta; \Gamma \vdash_P ([\ell_i = \varsigma(x_i^{D_i})b_i^{(i \in I)}].\ell_j \Leftarrow \varsigma(y^D)b) : D : \langle \ell_k : \tau \rightarrow \psi \rangle} \text{(Val Update}_1\text{)}$$

Figure 6 The Equational Theory of Objects

<p>(Eval Beta) :</p> $\frac{E \vdash_{\circ} \lambda x^A b : A \rightarrow B \quad E \vdash_{\circ} a : A}{E \vdash_{\top} (\lambda x^A b)(a) \leftrightarrow b\{x \leftarrow a\} : B}$	<p>(Eq Subsumption) :</p> $\frac{E \vdash_{\top} a \leftrightarrow a' : A \quad E \vdash_{\top} A <: B}{E \vdash_{\top} a \leftrightarrow a' : B}$	<p>(Eq Top) :</p> $\frac{E \vdash_{\circ} a : A \quad E \vdash_{\circ} b : B}{E \vdash_{\top} a \leftrightarrow b : \text{Top}}$
<p>(Eval Select) where $I \cap J = \emptyset$, $A \equiv [\ell_i : B_i \text{ }^{i \in I}]$, $A' \equiv [\ell_i : B_i \text{ }^{i \in I \cup J}]$, $a \equiv [\ell_i = \varsigma(x_i^{A'}) b_i \text{ }^{i \in I}]$:</p> $\frac{E \vdash_{\circ} a : A}{E \vdash_{\top} a.l_j \leftrightarrow b_j\{x_j \leftarrow a\} : B_j} \quad (j \in I)$		
<p>(Eval Update) where $I \cap J = \emptyset$, $A \equiv [\ell_i : B_i \text{ }^{i \in I}]$, $A' \equiv [\ell_i : B_i \text{ }^{i \in I \cup J}]$, $a \equiv [\ell_i = \varsigma(x_i^{A'}) b_i \text{ }^{i \in I}]$:</p> $\frac{E \vdash_{\circ} a : A \quad E, x:A \vdash_{\circ} b : B_j}{E \vdash_{\top} a.l_j \Leftarrow \varsigma(x^A)b \leftrightarrow [\ell_j = \varsigma(x^{A'})b, \ell_i = \varsigma(x^{A'})b_i \text{ }^{(i \in I \cup J \setminus \{j\})}] : A} \quad (j \in J)$		
<p>(Abs Cong) :</p> $\frac{E, x:A \vdash_{\top} b \leftrightarrow b' : B}{E \vdash_{\top} \lambda x^A . b \leftrightarrow \lambda x^A . b' : A \rightarrow B}$	<p>(App Cong) :</p> $\frac{E \vdash_{\top} a \leftrightarrow a' : A \rightarrow B \quad E \vdash_{\top} b \leftrightarrow b' : A}{E \vdash_{\top} a(b) \leftrightarrow a'(b') : B}$	
<p>(Object Cong) $A \equiv [\ell_i : B_i \text{ }^{i \in I}]$:</p> $\frac{E, x_i : A \vdash_{\top} b_i \leftrightarrow b'_i : B_i \quad (\forall i \in I)}{E \vdash_{\top} [\ell_i = \varsigma(x_i^A) b_i \text{ }^{i \in I}] \leftrightarrow [\ell_i = \varsigma(x_i^A) b'_i \text{ }^{i \in I}] : A}$	<p>(Sel Cong) $A \equiv [\ell_i : B_i \text{ }^{i \in I}]$, $i \in I$:</p> $\frac{E \vdash_{\top} a \leftrightarrow a' : A}{E \vdash_{\top} a.l_i \leftrightarrow a'.l_i : B_i}$	
<p>(Update Cong) $A \equiv [\ell_i : B_i \text{ }^{i \in I}]$, $j \in I$:</p> $\frac{E \vdash_{\top} a \leftrightarrow a' : A \quad E, x:A \vdash_{\top} b \leftrightarrow b' : B_j}{E \vdash_{\top} a.l_j \Leftarrow \varsigma(x^A)b \leftrightarrow a'.l_j \Leftarrow \varsigma(x^A)b' : A}$		

If instead $j \neq k$, we use rule (Val Update₂):

$$\frac{\Delta; \Gamma, x_k : D : \tau \vdash_{\top} b_k : D_j : \psi \quad \widehat{\Delta}; \widehat{\Gamma}, x_i : D \vdash_{\circ} b_i : D_i \quad (\forall i \in I \setminus k)}{\frac{\Delta; \Gamma \vdash_{\top} [\ell_i = \varsigma(x_i^{D_i}) b_i \text{ }^{(i \in I)}] : D : \langle \ell_k : \tau \rightarrow \psi \rangle \quad \widehat{\Delta}; \widehat{\Gamma}, y : D \vdash_{\circ} b : D_j}{\Delta; \Gamma \vdash_{\top} ([\ell_i = \varsigma(x_i^{D_i}) b_i \text{ }^{(i \in I)}] . l_j \Leftarrow \varsigma(y^D) b) : D : \langle \ell_k : \tau \rightarrow \psi \rangle} \text{ (Val Update}_2\text{)}}}$$

and the result follows by applying rule (<:).

For $\tau = \omega$, the result follows from Lemma 3.3, Theorem 4.5 and rule (ω). For $\tau = \bigwedge_{i \in I} \tau_i$ ($n \geq 0$), the proof follows by easy induction. ■

5 The Logical Equivalence

In [1] an equational theory of the object calculus is presented, whose first order sub-theory is generated by the rules of Figure 6 (omitting the term folding-unfolding rules, which do not make sense for the Object Calculus we consider here).

Definition 5.1 THE EQUATIONAL THEORY OF OBJECTS. The *Equational Theory of Objects* is a theory of equations of the shape $a \leftrightarrow b : A$, where a and b are pre-terms and A is a pre-type; these are derived as the right =-hand side of sequents $E \vdash_T a \leftrightarrow b : A$, where E is a (pre)-context of the Object Calculus. The rules are given in Figure 6, plus the obvious rules for reflexivity, symmetry and transitivity of \leftrightarrow .

Remark 5.2 Such a notion of equality includes (typed) convertibility, as it is clear from rules (Eval Beta), (Eq Select) and (Eq Update), but it does not coincide with it: in fact ‘ \leftrightarrow ’ is a congruence whereas ‘ \longrightarrow ’ is not closed under arbitrary contexts; more importantly, this is a consequence of subtyping and precisely of rule (Eq Sub Object) (see Example 5.4).

The assignment system of Definition 3.2 induces a logical notion of equivalence, according to which a and b are equal at A if they can be assigned the same set of predicates σ such that $A:\sigma$. More precisely, and taking into account contexts and their extensions into bases, we can formalise this idea as follows:

Definition 5.3 LOGICAL EQUIVALENCE. Let a be any pre-term; we define $\mathcal{P}(E, a, A)$ as the set of predicates of type A that can be assigned to a when $E \vdash_O a : A$:

$$\mathcal{P}(E, a, A) = \{ \sigma \mid \exists \Delta; \Gamma [\widehat{\Delta}; \widehat{\Gamma} \equiv E \ \& \ \Delta; \Gamma \vdash_P a : A : \sigma] \}.$$

We then say that the pre-terms a and b are *logically equivalent* at A and context E if they can be assigned the same set of predicates of type A with respect to E :

$$a \simeq_E b : A \iff \mathcal{P}(E, a, A) = \mathcal{P}(E, b, A).$$

Example 5.4 As in Example 4.8, let $A \equiv [\ell_0:Int, \ell_1:Int]$, and $a \equiv [\ell_0 = \varsigma(x_0^A)1, \ell_1 = \varsigma(x_1^A)x.\ell_0]$. Further consider:

$$b \equiv [\ell_0 = \varsigma(x_1^A)1, \ell_1 = \varsigma(x_1^A)1].$$

In [1], Section 7.6.2 it is argued that they cannot be equated at A . Indeed, they are not logically equivalent at A since, if we assume that 1 is the predicate expressing the property of “being the number 1”, so $1 \in \mathcal{L}_{Int}$, and $\Delta; \vdash_P 1:Int : 1$, then $\Delta; \vdash_P b:A : \langle \ell_1:\omega \rightarrow 1 \rangle$ but $\Delta; \not\vdash_P a:A : \langle \ell_1:\omega \rightarrow 1 \rangle$. Indeed (omitting again the Δ ’s and all those parts of the derivation justifying the assignment of the predicate to a type):

$$\frac{\overline{x_1:A:\omega \vdash_P 1:Int : 1}}{\vdash_P b:A : \langle \ell_1:\omega \rightarrow 1 \rangle} \text{ (Val Object)}$$

Replacing b by a would not yield a valid derivation. The best we can do in the case of a is instead:

$$\frac{\frac{\overline{x_1:A:\langle \ell_0:\omega \rightarrow 1 \rangle} \vdash_P x_1:A : \langle \ell_0:\omega \rightarrow 1 \rangle \quad \overline{x_1:A:\langle \ell_0:\omega \rightarrow 1 \rangle} \vdash_P x_1:A : \omega}{\vdash_P a:A : \langle \ell_1:\langle \ell_0:\omega \rightarrow 1 \rangle \rightarrow 1 \rangle} \text{ (Val Select)}}{\vdash_P a:A : \langle \ell_1:\langle \ell_0:\omega \rightarrow 1 \rangle \rightarrow 1 \rangle} \text{ (Val Object)}$$

To express this in natural language, what we have proven is that the value of b on calling method ℓ_1 is 1, and that this is a “field”, in that it does not depend on other parts of b ; on the other hand, for a the value returned by ℓ_1 depends on the actual value of ℓ_0 : the predicate $\langle \ell_1:\langle \ell_0:\omega \rightarrow 1 \rangle \rightarrow 1 \rangle$ expresses this.

However, in [1] paragraph 8.4.2 it is observed that the equality $\vdash_T a \leftrightarrow b : [\ell_0:Int]$ is derivable since both

$$\vdash_T [\ell_0 = \varsigma(x_0^B)1] \leftrightarrow a : [\ell_0:Int] \text{ and } \vdash_T [\ell_0 = \varsigma(x_0^B)1] \leftrightarrow b : [\ell_0:Int]$$

can be obtained by rule (Eq Sub Object); this clearly shows that ‘ \leftrightarrow ’ is not convertibility, since a , b and $[\ell_0 = \varsigma(x_0^B)1]$ are distinct normal forms and the reduction is confluent.

In our setting, we can show that $a \simeq_{\emptyset} b : [\ell_0:Int]$ as well, and this is the effect of restricting to the language $\mathcal{L}_{[\ell_0:Int]}$: in fact the only non-trivial predicates in $\mathcal{L}_{[\ell_0:Int]}$ that we can derive for either a or b are $\langle \ell_0:\omega \rightarrow 1 \rangle$ (or greater than this w.r.t. \leq).

We relate here logical equivalence to the equational theory of the Object Calculus.

Lemma 5.5 *Logic equivalence is a congruence; more precisely:*

$$a \simeq_{E,E'} a' : A \ \& \ b \simeq_{E,x:A,E'} b' : B \ \Rightarrow \ b\{x \leftarrow a\} \simeq_{E,E'} b'\{x \leftarrow a'\} : B.$$

Proof: This is just a rephrasing of the substitution property stated in Lemma 4.6. ■

We want to establish that equality in the Object Calculus implies logical equivalence, proving that what we have seen in the Example 5.4 actually holds in general. Corollary 2.17 is a first evidence of the consistency of the predicate assignment system with respect to the subtyping relation. It is however not enough, and we need Lemma 3.5.

Theorem 5.6 *If $E \vdash_{\top} a \leftrightarrow b : A$ then $a \simeq_E b : A$.*

Proof: The proof is by induction on the derivation of $E \vdash_{\top} a \leftrightarrow b : A$. In case the derivation ends by any rule among (Eval Beta), (Eval Select) and (Eval Update) the thesis follows by Theorems 4.7 and 4.10. Those cases in which the derivation ends by rules that establish the congruence properties of \leftrightarrow , namely (Abs Cong), (App Cong), (Object Cong), (Sel Cong) and (Update Cong), the result follows from Lemma 5.5. If the last rule is (Eq Top) then $A \equiv \text{Top}$ and we observe that, by Lemma 2.10, $\mathcal{P}(E, a, \text{Top})$ and $\mathcal{P}(E, b, \text{Top})$ contain exactly all predicates equivalent to ω , so they coincide.

The case remains in which the derivation ends by rule (Eq Subsumption):

$$\frac{E \vdash_{\top} a \leftrightarrow b : A \quad E \vdash_{\top} A <: B}{E \vdash_{\top} a \leftrightarrow b : B}$$

Let $\tau \in \mathcal{P}(E, a, B)$: then $\Delta; \Gamma \vdash_{\top} a : B : \tau$ for some $\Delta; \Gamma$ such that $\widehat{\Delta}; \widehat{\Gamma} \equiv E$. Since the premise $E \vdash_{\top} a \leftrightarrow b : A$ implies that both $E \vdash_{\top} a : A$ and $E \vdash_{\top} b : A$ are derivable and $E \vdash_{\top} A <: B$ implies that $\widehat{\Delta} \vdash_{\top} A <: B$, Lemma 3.5 applies, so that $\Delta; \Gamma \vdash_{\top} a : A : \tau$ is derivable, so $\tau \in \mathcal{P}(E, a, A)$. Also, by induction, $\mathcal{P}(E, a, A) = \mathcal{P}(E, b, A)$, that is $\Delta'; \Gamma' \vdash_{\top} b : A : \tau$ is derivable for certain $\Delta'; \Gamma'$ such that $\widehat{\Delta}'; \widehat{\Gamma}' \equiv E$. Then $\Delta'; \Gamma' \vdash_{\top} b : B : \tau$ follows by rule ($<:$), that is $\tau \in \mathcal{P}(E, b, B)$ as desired. ■

6 Logical Equivalence and Observational Semantics

Observational semantics for the first order calculus has been defined in [17] in Morris-style, called there “contextual equivalence”. It consists of *inseparability* by means of contexts of ground type. In the same paper it has been shown that this coincides with a notion of bi-simulation which is stronger than ‘ \leftrightarrow ’. We will adopt a slightly more general definition here.

We claim that, when restricted to closed terms, logical equivalence is included in observational equivalence. To this aim we will establish a computational adequacy result for the logical semantics w.r.t. convergence, which states that any well-typed term can be assigned a non-trivial predicate if and only if it converges to a value. This is achieved by means of a realizability interpretation of predicates, proving that the characterisation results of [15] are preserved in the typed context of the first order object calculus.

Definition 6.1 CONVERGENCE. Given any closed term a^A we say that it *converges* to the value v , written $a \Downarrow v$, if $a \xrightarrow{*} v$. Moreover we say that a^A is *convergent*, written $a \Downarrow$, if there exists a value v such that $a \Downarrow v$.

We will write $_ : A \vdash_{\circ} C[_] : B$ to express that the closed context $C[_]$ is typed with B , under the assumption that the “hole $_$ ” has type A ; $C[a]$ is the result of replacing ‘ $_$ ’ by a in $C[_]$.

Definition 6.2 OBSERVATIONAL EQUIVALENCE. Two closed terms a and b are called *observationally equivalent at type A* , written $a \simeq_A^O b$, if both a^A and b^A , and

$$\forall C[_]. _ : A \vdash C[_] : K \Rightarrow (C[a] \Downarrow v \iff C[b] \Downarrow v).$$

for any ground type K and value v of type K .

Remark 6.3 i) Typeable values can always be assigned non-trivial predicates.

ii) Definition 6.2 differs from the definition of contextual equivalence in [17] in some respect. First, we consider contexts of any ground type as an “experiment”; moreover, we do not consider reduction rules for constants as “if then else”; as a consequence we cannot discriminate between different constants like **true** and **false**. It is for that reason that we use in the above definition and in Theorem 6.12 the predicate $a \Downarrow v$ instead of $a \Downarrow$.

In the following we will write a^A for a closed term a of a closed type A , i.e such that $\emptyset \vdash_{\circ} a : A$ (abbreviated by $\vdash_{\circ} a : A$). In the next definition, the set of labels of A is defined as: $Label(A) = \{\ell_i \mid i \in I\}$ only for $A \equiv [\ell_i : A_i \ (i \in I)]$; it is empty in all other cases. If a^A for some object type A , $\ell_j \in Label(A)$ and $a \Downarrow [\ell_i = \varsigma(x_i^A) b_i \ (i \in I)]$, then $a.\ell(c)$ abbreviates $b_j \{x_j \leftarrow c\}$, for any c^A .

Definition 6.4 REALIZABILITY INTERPRETATION. The *realizability interpretation* of the predicate σ is a set $((\sigma))$ of closed terms defined by induction on the structure of predicates as follows:

- i) $((\omega)) = \{a^A \mid A \text{ is a closed type}\}$,
- ii) $((\sigma \wedge \tau)) = ((\sigma)) \cap ((\tau))$,
- iii) $((\kappa)) = \{a^K \mid \kappa \in \mathcal{L}_K \ \& \ \exists v [\vdash_{\circ} v : K : \kappa \ \& \ a \Downarrow v]\}$,
- iv) $((\sigma \rightarrow \phi)) = \{a^{A \rightarrow B} \mid \exists x, b [a \Downarrow (\lambda x^A. b) \ \& \ \forall c^A \in ((\sigma)) [b \{x \leftarrow c\} \in ((\phi))]]\}$,
- v) $((\langle \ell : \phi \rangle))$ is defined according to the shapes of ϕ :
 - $((\langle \ell : \omega \rangle)) = \{a^A \mid \ell \in Label(A) \ \& \ a \Downarrow\}$
 - $((\langle \ell : \sigma \rightarrow \psi \rangle)) = \{a^A \mid \ell \in Label(A) \ \& \ a \Downarrow \ \& \ \forall c^A \in ((\sigma)) [a.\ell(c) \in ((\psi))]\}$,
 - $((\langle \ell : \langle \ell' : \psi \rangle \rangle)) = \emptyset$.

The clause $((\langle \ell : \langle \ell' : \psi \rangle \rangle)) = \emptyset$ is consistent with the fact that $\langle \ell : \langle \ell' : \psi \rangle \rangle$ cannot be assigned to any (well typed) term.

Lemma 6.5 If $\sigma \neq \omega$ (i.e. it is non trivial), then any $a^A \in ((\sigma))$ converges.

Proof: By induction on the definition of $((\sigma))$. ■

Lemma 6.6 If $a^A \in ((\sigma))$ then for any b^A if $a \xrightarrow{*} b$ or $b \xrightarrow{*} a$ then $b^A \in ((\sigma))$.

Proof: By induction on the length of the reduction sequence, of which we only show the proof for the relation $\xrightarrow{*}$. This is proven by induction on the definition of reduction and by cases on σ , of which we show one case:

$(a \equiv (\lambda x^C a') a'' \text{ and } \sigma \equiv \tau \rightarrow \psi) : \text{ then } (\lambda x^C a') a'' \Downarrow v \text{ for some } v \text{ (an abstraction) with certain properties; but this is true if and only if } b \equiv a' \{x \leftarrow a''\} \Downarrow v, \text{ since the reduction is deterministic; hence } b \in ((\tau \rightarrow \phi)) \text{ if and only if } a \in ((\tau \rightarrow \phi)).$ ■

Lemma 6.7 If $\sigma \leq \tau$ then $((\sigma)) \subseteq ((\tau))$.

Proof: By easy induction on the definition of \leq using Lemma 2.4. E.g. suppose $\langle \ell : \sigma \rightarrow \phi \rangle \leq \langle \ell : \tau \rightarrow \psi \rangle$ because $\tau \leq \sigma$ and $\phi \leq \psi$. By Definition 6.4, for any $a^A \in ((\langle \ell : \sigma \rightarrow \phi \rangle))$ we have $a \Downarrow$ and $\ell \in Label(A)$;

suppose that $c^A \in ((\tau))$: by induction $c^A \in ((\sigma))$, so that by hypothesis $a.\ell(c) \in ((\phi))$. The thesis follows since $((\phi)) \subseteq ((\psi))$ again by induction. \blacksquare

Let $a\{x_j \leftarrow b_j\}_{j \leq k}$ abbreviate the simultaneous substitution of b_j for x_j in a for all $1 \leq j \leq k$, and similarly $A\{X_i \leftarrow B_i\}_{i \leq h}$, substituting each B_i for X_i in A for all $1 \leq i \leq h$.

Definition 6.8 COMPATIBLE SUBSTITUTIONS. Assume $\Delta; \Gamma \vdash_P a:A:\sigma$, where $\Delta \equiv J_1, \dots, J_h$ and each judgement J_i either is $X_i:\sigma_i$ or $X_i:\sigma_i <: D_i$, and $\Gamma \equiv x_1:C_1:\tau_1, \dots, x_k:C_k:\tau_k$. We say that the simultaneous substitutions $\{X_i \leftarrow B_i\}_{i \leq h}$ and $\{x_j \leftarrow b_j\}_{j \leq k}$ are *compatible* with $\Delta; \Gamma$ if: $\widehat{\Delta} \vdash_{\top} B_i <: D_i$ (in case $X_i:\sigma_i <: D_i \in \Delta$), and $\vdash_0 B_i:\sigma_i$ for all $1 \leq i \leq h$, and $\vdash_P b_j:C_j\{X_i \leftarrow B_i\}_{i \leq h}:\tau_j$ for all $1 \leq j \leq k$.

In the above definition, if $\Delta; \Gamma \vdash_P a:A:\sigma$ is derivable then $fv(A) \cup fv(a) \subseteq dom(\Delta) \cup dom(\Gamma)$: hence compatible substitutions are closing a and A if they replace all free variable occurrences by closed types and terms. We will call such a substitution a *closing* substitution.

Lemma 6.9 Assume that the closing substitutions $\{X_i \leftarrow B_i\}_{i \leq h}$ and $\{x_j \leftarrow b_j\}_{j \leq k}$ are compatible with $\Delta; \Gamma$, then:

$$\text{if } \Delta; \Gamma \vdash_P a:A:\sigma \text{ then } \vdash_P a\{x_j \leftarrow b_j\}_{j \leq k}:A\{X_i \leftarrow B_i\}_{i \leq h}:\sigma$$

Proof: By Lemma 2.14 and Lemma 4.6. \blacksquare

Theorem 6.10 REALIZABILITY THEOREM. Let $\Delta; \Gamma \vdash_P a:A:\sigma$; assume that the closing substitutions $\{X_i \leftarrow B_i\}_{i \leq h}$ and $\{x_j \leftarrow b_j\}_{j \leq k}$ are compatible with $\Delta; \Gamma$. If $b_j^{C'_j} \in ((\tau_j))$ whenever $x_j:C_j:\tau_j \in \Gamma$, then $a^{A'} \in ((\sigma))$, where $a' \equiv a\{x_j \leftarrow b_j\}_{j \leq k}$, $C'_j \equiv C_j\{X_i \leftarrow B_i\}_{i \leq h}$ and $A' \equiv A\{X_i \leftarrow B_i\}_{i \leq h}$.

Proof: We write $a\vartheta$ for $a\{x_j \leftarrow b_j\}_{j \leq k}$ and $A' \equiv A\Theta$ for $A\{X_i \leftarrow B_i\}_{i \leq h}$. The proof proceeds by induction on the derivation for $\Delta; \Gamma \vdash_P a:A:\sigma$. The case (Val x) is trivial; the case ($<$) follows by the induction hypothesis and Lemma 6.7. For the remaining cases:

(Val Fun) : the last inference is an instance of the rule:

$$\frac{\Delta; \Gamma, x:A:\sigma \vdash_P a:B:\phi}{\Delta; \Gamma \vdash_P \lambda x^A.a:A \rightarrow B:\sigma \rightarrow \phi}$$

$(\lambda x^A.a)\vartheta$ is a value, hence it converges trivially. Since ϑ, Θ are compatible with $\Delta; \Gamma$, given any $b^{A'} \in ((\sigma))$ (where $A' \equiv A\Theta$), the substitutions $\vartheta[x := b]$ (which is the same as ϑ but for its value on x which is b) and Θ are compatible with $\Delta; \Gamma, x:A:\sigma$. Then, by induction $a\vartheta[x := b] \in ((\phi))$, and we conclude from $((\lambda x^A.a)\vartheta)b \longrightarrow a\vartheta[x := b]$, Lemma 6.6 and the arbitrary choice of $b^{A'}$.

(Val Appl) : the last inference is an instance of the rule:

$$\frac{\Delta; \Gamma \vdash_P a:A \rightarrow B:\sigma \rightarrow \phi \quad \Delta; \Gamma \vdash_P b:A:\sigma}{\Delta; \Gamma \vdash_P ab:B:\phi}$$

By induction $a\vartheta \in ((\sigma \rightarrow \phi))$ so that $a\vartheta \Downarrow (\lambda x^A.d)$ for some closed abstraction $\lambda x^A.d$; moreover $b\vartheta \in ((\sigma))$ by induction, hence $d[x := b\vartheta] \in ((\phi))$, and we conclude by Lemma 6.6 and

$$(ab)\vartheta \equiv (a\vartheta)(b\vartheta) \xrightarrow{*} (\lambda x^A.d)(b\vartheta) \longrightarrow d[x := b\vartheta].$$

(Val Object) : the last inference is an instance of the rule:

$$\frac{\Delta; \Gamma, x_j:A:\sigma \vdash_P b_j:B_j:\phi \quad \widehat{\Delta}; \overline{\Gamma}, x_i:A \vdash_0 b_i:B_i \quad (\forall i \in I \setminus \{j\})}{\Delta; \Gamma \vdash_P [\ell_i = \varsigma(x_i^A)b_i \ (i \in I)]:A:\langle \ell_j:\sigma \rightarrow \phi \rangle} \quad (j \in I)$$

Now $a \equiv [\ell_i = \varsigma(x_i^A)b_i \ (i \in I)]$ is an object term, whose closure is a value: hence $a\vartheta \Downarrow$. That $\ell_j \in \text{Label}(A)$ is a side condition of the rule. For any $d^{A'} \in ((\sigma))$ we have that $\vartheta[x_j := d]$, and Θ are compatible with $\Delta; \Gamma, x_j:A:\sigma$ and

$$a\vartheta.l_j(d) \equiv b(\vartheta[x_j := d]) \in ((\phi_j))$$

by induction.

(Val Select) : the last inference is an instance of the rule:

$$\frac{\Delta; \Gamma \vdash_P a:A : \langle \ell_j:\sigma \rightarrow \phi \rangle \quad \Delta; \Gamma \vdash_P a:A : \sigma}{\Delta; \Gamma \vdash_P a.l_j:B_j : \phi}$$

By induction $a\vartheta \Downarrow v$, for some value v , $\ell_j \in \text{Label}(A)$ and $a\vartheta.l_j(d) \in ((\phi))$ for any $d^{A'} \in ((\sigma))$; since $a\vartheta \in ((\sigma))$ (by induction again) we have that $v \in ((\sigma))$ by Lemma 6.6 so that:

$$(a.l_j)\vartheta \xrightarrow{*} a\vartheta.l_j(v) \in ((\phi)),$$

and we conclude again by Lemma 6.6.

(Val Update₁) : the last inference has the shape:

$$\frac{\Delta; \Gamma \vdash_P a:A : \tau \quad \Delta; \Gamma, y:A:\sigma \vdash_P b:B_j : \phi}{\Delta; \Gamma \vdash_P (a.l_j \Leftarrow \varsigma(y^A)b):A : \langle \ell_j:\sigma \rightarrow \phi \rangle} \ (\ell_j \in \text{Label}(A), \tau \neq \omega)$$

By induction $a\vartheta \in ((\tau))$; since $\tau \neq \omega$ this implies that $a\vartheta \Downarrow$ and that $\ell_j \in \text{Label}(A)$, therefore $(a.l_j \Leftarrow \varsigma(y^A)b)\vartheta \Downarrow$ as well. Given any $d^{A'} \in [[\sigma]]$, $\vartheta[y := d]$, Θ is compatible with $\Delta; \Gamma, y:A:\sigma$, so that we conclude by induction

$$(a.l_j \Leftarrow \varsigma(y^A)b)\vartheta.l_j(d) \equiv b\vartheta[y := d] \in ((\phi)).$$

(Val Update₂) : the last inference has the form:

$$\frac{\Delta; \Gamma \vdash_P a:A : \langle \ell_j:\phi \rangle \quad \widehat{\Delta; \Gamma}, y:A \vdash_P b:B_i}{\Delta; \Gamma \vdash_P (a.l_i \Leftarrow \varsigma(y^A)b):A : \langle \ell_j:\phi \rangle} \ (i \neq j)$$

By induction we know that $a\vartheta \in [[\langle \ell_j:\phi \rangle]]$, which implies that $a\vartheta \Downarrow$: hence $(a.l_j \Leftarrow \varsigma(y^A)b)\vartheta \Downarrow$. It remains to show that $(a.l_j \Leftarrow \varsigma(y^A)b)\vartheta \in ((\langle \ell_j:\phi \rangle))$ according to the shapes of ϕ . The case in which $\phi \equiv \omega$ is already proved by the fact that $(a.l_j \Leftarrow \varsigma(y^A)b)\vartheta \Downarrow$. The case $\phi \equiv \langle \ell:\psi \rangle$ is impossible, since $(a.l_j \Leftarrow \varsigma(y^A)b)\vartheta$ is well-typed because of the compatibility of ϑ , and no well-typed term might be assigned a predicate of that shape. We are left with the case $\phi \equiv \tau \rightarrow \psi$: then, since $i \neq j$, we have that $(a.l_j \Leftarrow \varsigma(y^A)b)\vartheta.l_i(d) \equiv a\vartheta.l_i(d)$ is in $((\psi))$ when $d^{A'} \in ((\tau))$ by the inductive hypothesis. ■

Corollary 6.11 CHARACTERISATION OF CONVERGENCE. *Let a^A be any closed term: then $a \Downarrow$ if and only if $\vdash_P a:A : \sigma$ for some non-trivial σ .*

Proof: As noted in (i) of Remark 6.3 typeable value v can be assigned non-trivial predicates, so that $a \Downarrow v$ implies that the same predicates can be derived for a because of Theorem 4.10; on the other hand a straightforward induction on the structure of σ shows that if σ is non trivial, then any $a^A \in ((\sigma))$ converges: by this and Theorem 6.10 we conclude. ■

Theorem 6.12 LOGICAL EQUIVALENCE AND OBSERVATIONAL EQUIVALENCE. *Suppose that for any value v of ground type K we have exactly a non-trivial predicate $\kappa_v \in \mathcal{L}_K$, that this predicates are distinct for different values and that $\vdash_P v:K : \kappa_v$ is assumed for each v . Then for any a^A and b^A , if $a \simeq b : A$ then $a \simeq_A^O b$.*

Proof: Let $_ : A \vdash C[_] : K$ be any context of ground type K such that $C[a] \Downarrow v$ for some value v . By the hypothesis that $\vdash_P v : K : \kappa_v$ and Theorem 4.10, we have $\kappa_v \in \mathcal{P}(\emptyset, C[a], K)$. On the other hand $a \simeq b : A$ implies $C[a] \simeq C[b] : K$, since logical equivalence is a congruence by Lemma 5.5, and therefore $\kappa_v \in \mathcal{P}(\emptyset, C[b], K)$, that is $\vdash_P C[b] : K : \kappa_v$. By Corollary 6.11 it follows that $C[b] \Downarrow v'$ for some v' , whence $\vdash_P v' : K : \kappa_v$ by Theorem 4.7. From the hypothesis that K is a ground type and that κ_v is a characteristic predicate of v , we conclude $v' \equiv v$. \blacksquare

By using bi-simulation and its coincidence with observational equivalence, in [17] it is shown that, taking a and b as in example 5.4, $a \simeq_{[\ell_1:Int]}^{\mathcal{O}} b$. This is intuitively clear: the only way to separate a from b is to change the value of ℓ_0 , since then the fact that $a.\ell_1$ depends on such a value while $b.\ell_1$ does not, becomes apparent; but the overriding of ℓ_0 is inhibited in contexts with the hole of type $[\ell_1:Int]$, where ℓ_0 is hidden.

It is not true, however, that $a \simeq b : [\ell_1:Int]$, because the predicate $\langle \ell_1 : \omega \rightarrow 1 \rangle$ is in $\mathcal{L}_{[\ell_1:Int]}$, it is derivable for b even at type $[\ell_1:Int]$ but cannot be derived for a at any type.

Language inclusion alone is not sufficient to account for subtyping of object types, while it is for record types (see [16]): this is the essential reason for the presence of rule (Val Select) in our system. It is reasonable to think that the failure of equivalences like $a \simeq b : [\ell_1:Int]$ from Example 5.4 depends on the fact that no rule accounts for the hiding effect of subtyping in the case of object types. One possibility for coping with such a limitation is the following rule:

$$\frac{I \cap J = \emptyset, A \equiv [\ell_i : B_i \text{ }^{i \in I \cup J}], A' \equiv [\ell_i : B_i \text{ }^{i \in J}] : \quad \Delta; \Gamma \vdash_P a : A : \langle \ell : \langle \ell_i : \sigma_i \text{ }^{i \in I} \rangle \wedge \tau \rightarrow \rho \rangle \quad \Delta; \Gamma \vdash_P a : A : \langle \ell_i : \sigma_i \text{ }^{i \in I} \rangle \quad \Delta \vdash_O A' : \langle \ell : \tau \rightarrow \rho \rangle}{\Delta; \Gamma \vdash_P a : A' : \langle \ell : \tau \rightarrow \rho \rangle}$$

This rule formalises the idea that when $A < A'$ and A and A' are object types, the methods of any object of type A not mentioned in A' are hidden: therefore if a satisfies the premise of any arrow predicate concerning the hidden part, this will never change in contexts of type A' , in such a way that the latter premise can be discharged. Clearly, with reference to Example 5.4, by this rule one can derive $\vdash_O b : [\ell_1:Int] : \langle \ell_1 : \omega \rightarrow 1 \rangle$, which makes a and b logically indiscernible at type $[\ell_1 : Int]$.

We have not considered this rule in our assignment system, however. The proof of the soundness of such a rule requires a different definition of the realizability interpretation, and makes the proof theory of the assignment system more involved. On the other hand, it is difficult to say to what extent we obtain a stronger equivalence. Indeed we know in advance that logical equivalence cannot coincide with observational equivalence: the former is indeed the theory of a filter model which is a D_∞ model; we know from [5] that it is not fully abstract w.r.t. the lazy λ -calculus, which is a sub-calculus of the Object Calculus we consider. Were logical equivalence and observational equivalence the same, a full abstraction property would hold for the model.

7 Conclusions and Related Work

The system and results presented in this paper have been developed through a series of papers, [15], [16], [9], and [10], of which the present work is an extended and revised version.

To summarise our work, it can be seen as an intersection type assignment system (see e.g. [10] and the references there) to typed terms of the first order object calculus from [1]. A similar idea of assignment of logical formulas to terms of a simply typed λ -calculus with recursive types is the endogenous logic in [4], where languages are used to provide a finitary description of the domain interpretation of types, and a denotation of terms, much as it happens for filter models of the type free λ -calculus in [11]. With respect to these antecedents our technical contribution consists both in the consideration of object types and, more importantly, in the treatment of subtyping, for which we build on ideas presented in [16].

Our results substantiate the claim that logical semantics allows for a clean and elegant understanding of subtyping, which is seen as a form of (reverse) inclusion of logical theories, indexed by types. This seems remarkable in presence of arrow, object and recursive types, whose combination is notoriously very difficult to model.

The resulting logical equivalence is consistent with the equivalence axiomatically defined in [1], and operationally sound with respect to the Morris style semantics studied in [17]. We remark that logic adds to the abstraction represented by types, in that it is able to capture computational properties like convergence and context separability, which is not the case for types. To show this, we resort to the technical tool of realizability interpretation of predicates, which comes from [15] and is a mild extension of known techniques from the λ -calculus. Its relation to types and subtyping, however, is not completely understood and is clearly involved in the treatment of the rule suggested at the end of Section 6.

The present study rests on the assignment system and its proof theoretical properties, without facing the problem of models. In [16] a filter model construction is proposed in which types are interpreted as the CUPERs (studied in [6, 3, 14] and used in [1] Ch. 14) induced by the indiscernibility relation w.r.t. the predicates of a language. Terms are interpreted as filters of predicates, but their meaning in a type A is obtained by restricting to \mathcal{L}_A . Such a restriction, trivially idempotent, is also continuous, which suggested the interpretation of the restriction operation in terms of retractions over a D_∞ universal model in [9]. Unfortunately retractions do not allow for a sound treatment of subtyping because of their covariant behaviour w.r.t. both left and right-hand sides of arrow types. Although we think that a model is implicitly described by our system, the analysis of its structure and the comparison with existing denotational models of subtyping deserve further investigation.

Our system provides a program logic for the first order ζ -calculus, which is natural to compare with similar proposals in the literature. In [2] a Hoare style logic for a first order object calculus with subtyping is presented. A close relationship exists between their transition relations and our predicates $\langle \ell_i: \sigma_i \rightarrow \phi_i \ (i \in I) \rangle$. In fact, even if transition relations are expressed via first order predicate logic whereas our logic is propositional, they specify pre and post conditions of methods in terms of properties of filed values before and after method invocation. We can do the same, since fields are simply methods that do not depend on self variables, so that we can encode their properties by means of predicates of the shape $\langle \ell: \omega \rightarrow \psi \rangle$ (and conjunctions of them); we then put them as the premises σ_i above, and encode post conditions in the ϕ_i . We stress however that our framework is more powerful, because in [2] only field update is permitted, whereas our logic is sound in the presence of the stronger operation of method overriding.

The latter limitation is removed in [19], at the price of losing monotonicity of the transition relations. To handle this difficulty, a notion of specification $Spec(A, \vec{B}, \vec{T})$ is introduced as the unique fixed point of the bi-functor $\Phi_{A, \vec{B}, \vec{T}}$ induced by the field predicate A , the result predicates \vec{B} and the transition relations \vec{T} ; such a fixed point does not exist in general, and it can be assured only under suitable conditions (Existence Theorem). We first remark a tight similarity between the definition of $Spec(A, \vec{B}, \vec{T})$ and of $((\langle \ell: \sigma \rightarrow \psi \rangle))$ here, especially for the quantification in the clause $\forall c^A \in ((\sigma)). a.\ell(c) \in ((\psi))$. Then we observe that the realizability interpretation of predicates is inductive, so that $((\sigma))$ always exists. Although we do not have a definite answer, it seems reasonable to think that our predicates are particular cases of specifications, enjoying the good properties, which would explain why we do not need an existence theorem at all. The model of the logic proposed in [19] is for the untyped ζ -calculus, both functional and imperative, so that there is nothing to remark about subtyping semantics here.

A different approach to the relationship between program logic and subtyping is “behavioural subtyping” as exposed e.g. in [20]. It is based on the “subtype requirement” which says that if $A <: B$ and $\phi(b)$ for all $b: B$ then $\phi(a)$ for all $a: A$, where $\phi(x)$ is a certain predicate of x . This recalls the fact that $\mathcal{L}_A \supseteq \mathcal{L}_B$ whenever $A <: B$ in our system. However, because of the universal quantifiers in the subtyping requirement, the predicates of [20] are likely to be properties of types rather than of programs, so that they are better seen as a reinforcement of the abstractions expressed by types, rather

than as a description of behaviours. The latter is exactly what we gain in our system, as shown via the Realizability Theorem 6.10.

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