An Approximate Compositional Approach to the Analysis of Fluid Queue Networks

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Abstract

Fluid models have for some time been used to approximate stochastic networks with discrete state. These range from traditional 'heavy traffic' approximations to the recent advances in bio-chemical system models. Here we present a simple approximate compositional method for analysing a network of fluid queues with Markov-modulated input processes at equilibrium. The idea is to approximate the *on/off* process at the output of a queue by an n-state Markov chain that modulates its rate. This chain is parameterised by matching the moments of the resulting process with those of the busy period distribution of the queue. This process is then used, in turn, as a separate Markov-modulated *on/off* process that feeds downstream queue(s). The moments of the busy period are derived from an exact analytical model. Approximation using two- and three-state intermediate Markov processes are validated with respect to an exact model of a tandem pair of fluid queues — a generalisation of the single queue model. The analytical models used are rather simpler and more accessible, albeit less general, than previously published models, and are also included. The approximation method is applied to various fluid queue networks and the results are validated with respect to simulation. The results show the three-state model to yield excellent approximations for mean fluid levels, even under high load.

Key words: Fluid queue, network, *on/off* process, analytical model, approximation

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1 Introduction

Stochastic fluid flow models have for long been used to describe networks of nodes that provide service to traffic of some sort that flows amongst them. Such models can exactly describe systems with *continuous state*, for example volumes in literal fluid flows, but more commonly are used to approximate *discrete state systems* of *traffic flows*. Traffic is measured in integer units of work such as packets in an IP network, jobs in a computer system or vehicles on roads. The motivation for these continuous approximations is that the numbers of states in discrete systems rapidly become prohibitively large as the complexity of the systems increase. Moreover, it can often be proved that as the rates of the discrete traffic inputs and of processing at the nodes jointly tend to infinity, the system's behaviour approaches that of a corresponding continuous model – a so-called 'fluid limit' [1]. Of course, in practice it is not a pre-requisite for such a limit to exist when entertaining a fluid-based model since all models are just abstractions of a real system. It may be that no analogous discrete model has been considered at all or, if it has, for that model to be inherently superior.

The adoption of fluid models has recently been taken up in the field of stochastic process algebra where large numbers of cooperating components in a concurrent system are approximated by a 'volume' of fluid [2]. Process algebras are notorious for their general profligacy in state space, but recent applications in biochemistry increase traditional discrete state space sizes by several orders of magnitude. Hence the representation of a very large number of cooperating identical components by a single non-negative real number is attractive and has met with considerable success [3].

The single fluid queue has been studied in some depth and results exist under quite general assumptions about the input processes—see for example [4]. Various flavours of tandem queues, fed by *on/off* processes have also been analysed to obtain the steady-state joint distribution of the fluid levels in each queue, from which various measures can be derived, e.g. [5–7]. Similar exact results for more general networks have also been produced using Martingale methods [8,9].

In this paper we evaluate a simple approximate approach to the analysis of steady-state fluid queue chains with a single *on/off* Markov modulated external arrival process. The approach generalises trivially to "tree-like" networks of queues by suitably scaling the output rate of a queue to reflect branching probabilities in the tree.

Our approach is motivated by the observation that the output process of each fluid queue in such a network is also an *on/off* process, although the busy

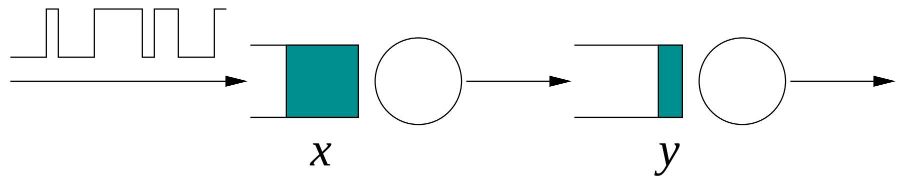

Fig. 1. A tandem pair of fluid queues with *On/Off* arrivals

periods will not be exponentially distributed. The idea is to approximate this *on/off* process by an n-state Markov chain that modulates its rate. One of the states of the chain reproduces the *off* state of the external arrival process; the *off*-periods of the process feeding an internal queue have the same probability distribution as the *off*-period of the external process when this is an exponential random variable. The sum of the times spent in the remaining $n-1$ states of the Markov chain (before returning to the one modelling the *off* state) seeks to approximate the busy period of the queue in question. The generators of the approximating Markov chain are estimated by matching the first k moments of the generated *on*-periods with the first k moments of the busy period being approximated. The latter comes from an exact analytical model of a single fluid queue. The value of k is at least the number of unknown generators in a standard least squares estimation; note that it is not always possible to find positive generators that match exactly an arbitrary set of given moments.

The accuracy of the approximation and the number of moments that need to be matched depends on n and on the structure of the chain. Here we focus on a simple two- and three-state Markov chains and use moment matching to determine the chains' unknown parameters (which number one and three respetively). The number of moments that can be matched exactly depends on the parameterisation of the network.

The analytical models that we used in the moment matching and validation are included as appendices. We could have used any of the previously published analyses, but have included ours partly for self-containedness and because they present an interesting, alternative approach in its own right. Our analysis is less general than some others, which tend to use powerful martingale methods, but it is arguably simpler and more accessible. It can be shown that, for the specific cases we consider, our model produces the same results as those in the literature.

Section 2 defines the systems that are the subject of the paper. A summary of our analytical model for a tandem pair of fluid queues is given in Section 3, with the bulk of the detail appearing in the Appendices. The approximation technique is detailed in Section 4, which includes a numerical evaluation of the accuracy of the approach for a small range of fluid queueing networks. The conclusions of the paper are given in Section 5.

2 Fluid Queueing Networks

We consider, at equilibrium, a linear chain of fluid queues, i.e. with no cycles, with a Markov modulated *on/off* arrival process (MMOAP) at the first queue as the only input source. The simplest non-trivial model is a tandem pair of queues, as illustrated in Figure 1, which is the subject of the model described in Section 3. The networks we consider in general have the following characteristics:

- A Markov modulated *on/off* arrival process at the leftmost (root) node, numbered 1, in the chain (tree), with generator matrix $Q = \begin{pmatrix} -a & a \\ b & -b \end{pmatrix}$ and equilibrium probabilities $\vec{\pi} = (b, a)/(a + b)$.
- Constant *on*-rate λ in state 1 and zero in state 2 for arrivals at node 1. There are no external arrivals elsewhere in the network.
- The root server outputs fluid at constant rate μ_1 when its reservoir is non-empty, sending it to the successor node (node 2 in a chain). No fluid is output from an empty reservoir.
- A similar argument applies to successor nodes. In these cases, however, the input process is defined by the output process at the preceding node, itself an *on/off* process.

In our approximate analysis, the single fluid queue at equilibrium is the building block.

In our approximate analysis, the single fluid queue at equilibrium is the building block. For each queue in the chain, the input process is a given k-state MMOAP, where $k = 2$ or 3 in our two approximations of section 4. The fluid input rate is zero in one state and equal to the specified rate in the other $k - 1$ states. The probability density function of its fluid level is calculated (the objective of the whole exercise) as well as the moments of its busy period. Sufficient moments are then used to parameterise the MMOAP for the successor node, by matching against those for the first passage time of the assumed modulating Markov chain from its *off* state and back, as discussed above.

The result for the fluid level density is well known but is included in Appendix A for completeness and since it forms the basis for our analysis of a tandem pair of queues. Using the notation defined there, the Laplace transform of the busy period's probability density function is as follows.

Proposition 1 *Consider an* on/off *fluid process with exponentially distributed* off *time, parameter b, and constant fluid rates as defined in Appendix A for the single fluid queue. Let the* on *period (respectively, busy-period) random variable be denoted by V (respectively W), with probability distribution, density* *function and Laplace transform* $V(t), v(t) = V'(t)$ *and* $V^*(\theta)$ *(respectively,* $W(t), w(t) = W'(t)$ *and* $W^*(\theta)$*). Then*

$$W^*(\theta) = V^*\Big(\theta\rho + b(\rho - 1)(1 - W^*(\theta))\Big)$$

where $\rho = \lambda/\mu > 1$*.*

Proof A busy period begins at an instant when the state of the MMOAP changes from *off* to *on* (and the fluid level is zero). At the end of the ensuing *on*-period, which has length V, the fluid level is $(\lambda-\mu)V$, which will eventually take $(\rho - 1)V$ time units to drain during subsequent *off*-periods.

As far as the busy period is concerned, it matters not which particle of fluid is the next to be output at any instant, compare the standard argument for the $M/G/1$ queue. We assume that fluid is output on a last-in-first-out basis. Then the time elapsed between the first state transition from *on* to *off* (after time V) until the fluid level is next $(\lambda - \mu)V$ (comprising the same particles as were in the reservoir after time V) has the same probability distribution as a busy period. Call this elapsed time W_1.

Repeating this argument, if the reservoir next becomes empty after C such pseudo-busy periods, we find

$$\begin{aligned} W &= V + (\rho - 1)V + W_1 + \ldots + W_C \\ &= \rho V + \sum_{i=1}^{C} W_i \end{aligned}$$

Hence,

$$\begin{aligned} W^*(\theta) &= E[e^{-\theta W}] \\ &= E[E[e^{-\theta(\rho V + \sum_{i=1}^{C} W_i)}|V]] \\ &= E[e^{-\theta\rho V} E[E[e^{-\theta W}|C, V]^C|V]] \\ &= E[e^{-\theta\rho V} G_C(W^*(\theta))] \end{aligned}$$

where G_C is the probability generating function (pgf) of C given V. By construction, the sum of the C *off*-periods preceding the C pseudo busy periods is less than $(\rho-1)V$; otherwise all the fluid would already have drained out. But $(\rho-1)V$ is less than this sum of the C *off*-periods added to the next one, since the fluid drains out during the $(i+1)$st. Therefore C has Poisson distribution with parameter $b(\rho-1)V$ and pgf $G_C(z) = e^{-b(\rho-1)V(1-z)}$. We therefore obtain

$$
\begin{aligned}
W^*(\theta) &= E[e^{-\theta\rho V} e^{-b(\rho-1)V(1-W^*(\theta))}] \\
&= V^*(\theta\rho + b(\rho-1)(1-W^*(\theta)))
\end{aligned}
$$

♠

A more general result of [4] determines the Laplace transform of the busy period density when there are multiple MMOAP arrival streams. The proof approach is the same with some subtle complications that require a more delicate treatment. This result could be needed to extend our methodology to feedforward networks, in which nodes may have multiple upstream sources. However, for the present study of tandem and treelike networks, the above proposition suffices.

The first three moments that we require are obtained by direct differentiation at $\theta = 0$.

Corollary 2 *The first tree moments B_1, B_2, B_3 of the busy period are:*

$$
\begin{aligned}
B_1 &= \frac{\rho O_1}{1 - b(\rho-1)O_1} \\
B_2 &= \frac{(\rho + b(\rho-1)B_1)^2 O_2}{1 - b(\rho-1)O_1} \\
B_3 &= \frac{(\rho + b(\rho-1)B_1)^3 O_3 + 3b(\rho-1)O_2 B_2}{1 - b(\rho-1)O_1}
\end{aligned}
$$

where O_1, O_2, O_3 are the first three moments of the on-*period.*

3 Analytical Model for Tandem Queues

The tandem pair of fluid queues is the subject of our analytical model, which in turn is used to validate our approximations. In particular, it shows that there is no separable solution. Consequently there would appear to be little hope for exact tractable solutions in large networks, akin to the product-forms of Markovian discrete queueing networks. In a tandem pair, server 2 receives fluid input at constant rate μ_1 when node 1 has a non-empty reservoir, has no input when node 1 has an empty reservoir and outputs fluid at constant rate μ_2 when its own reservoir is non-empty;

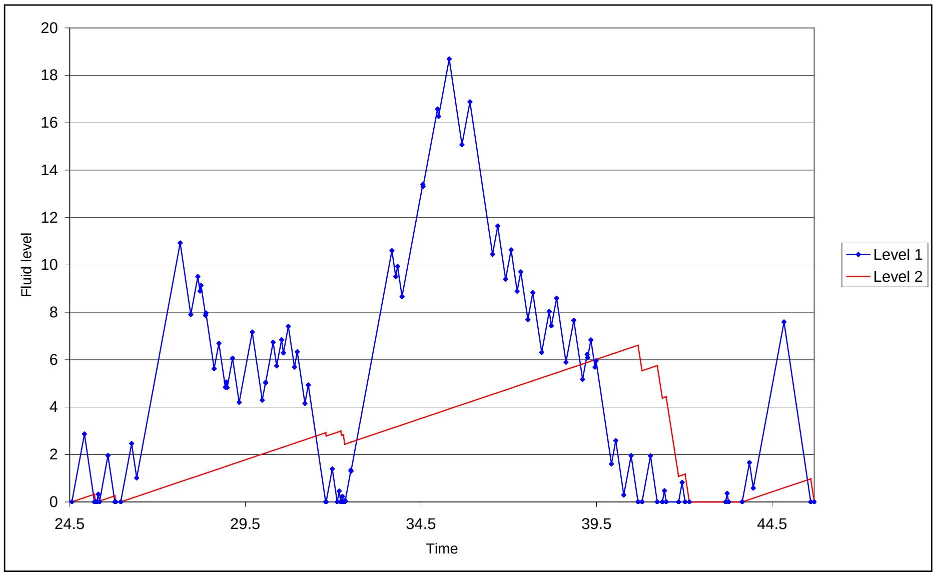

Fig. 2. Example sample paths ($a = b = 5, \lambda = 18, \mu_1 = 10, \mu_2 = 9.5$)

We define the diagonal matrices:

$$R = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}$$

where $r_1 = \lambda - \mu_1, r_2 = -\mu_1$ and $s_1 = s_2 = s = \mu_1 - \mu_2$, and denote by N_t, X_t and Y_t the state of the arrival process to node 1 and the fluid levels in reservoirs 1 and 2 at time t respectively.

Note that if $\lambda \leq \mu_1$ then trivially $X_t = 0$ for all $t \geq 0$; similarly for Y_t when $\mu_1 \leq \mu_2$. We therefore assume throughout that $\lambda > \mu_1 > \mu_2$. Under this assumption, X_t increases at rate r_1 when the modulated arrival process is in state 1 (on) and at rate $r_2 = -\mu_1$ when the arrival process is in state 2 (*off*) and $X_t > 0$. Y_t similarly increases at rate s when $X_t > 0$ and, at rate $-\mu_2$ when $Y_t > 0, X_t = 0$. A sample path of either of the processes X_t and $Y_t, t \geq 0$ therefore comprises periods where the fluid level is either zero, or rising/falling at the specified rate. The rising periods corresponds to the busy periods of the process that feeds the queue. Example sample paths for X_t (Level 1) and Y_t (Level 2) are shown in Figure 2 for $a = b = 5, \lambda = 18, \mu_1 = 10, \mu_2 = 9.5$.

Note that Y_t can be 0 only when $X_t = 0$. Consequently, the *off*-periods of the input to the second queue have the same probability distribution as at the first queue (exponential, rate b).

The vector differential equation describing two queues in tandem – and indeed more complex networks – is easily derived using the arguments used for a single queue in Appendix A. Define $\vec{F}(x,y,t) = (F_1(x,y,t), F_2(x,y,t))$, where

$$F_i(x,y,t) = P(N_t = i, X_t \leq x, Y_t \leq y)$$

Now consider the infinitesimal interval $(t, t+h]$ for some small h.

The interesting difference from the single fluid queue model is that when reservoir 1 is empty, there is no input to queue 2, and we have $F_1(0,y) = 0$ for all $y \geq 0$ since $r_1 > 0$ (as for the single queue) and, for $i = 2$ in particular,

$$F_i(0,y,t+h) = (1+q_{ii})F_i(-r_i h, y - s_i' h, t) + \sum_{j \neq i} F_j(x,y,t)q_{ji}h + o(h)$$

where $S' = \mathrm{diag}(-\mu_2, -\mu_2) = -\mu_2 I$.

This gives the boundary equation at $x = 0$:

$$\frac{\partial \vec{F}(0,y)}{\partial x}R + \frac{\partial \vec{F}(0,y)}{\partial y}S' - \vec{F}(0,y)Q = 0 \tag{1}$$

For $x, y > 0$, to first order in h, we have

$$\begin{aligned}
F_i(x,y,t+h) =& (1+q_{ii})\Big(P(0 < X_t \leq x - r_i h, Y_t \leq y - s_i h, t)s \\
&+ P(X_t = 0, Y_t \leq y - s_i' h, t)\Big) + \sum_{j \neq i} F_j(x,y,t)q_{ji}h + o(h) \\
=& (1+q_{ii})\Big(F_i(x - r_i h, y - s_i h, t) - F_i(0, y - s_i h, t) \\
&+ F_i(0, y - s_i' h, t)\Big) + \sum_{j \neq i} F_j(x,y,t)q_{ji}h + o(h)
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{F_i(x,y,t+h) - F_i(x,y,t)}{h} =& -r_i \frac{\partial F_i(x,y,t)}{\partial x} - s_i \frac{\partial F_i(x,y,t)}{\partial y} \\
&+ (s_i - s_i')\frac{\partial F_i(0,y,t)}{\partial y} + \sum_{j=1}^{n} F_j(x,y,t)q_{ji} + O(h)
\end{aligned}$$

so that in the limit $h \to 0$

$$\frac{\partial \vec{F}(x,y,t)}{\partial t} = -\frac{\partial \vec{F}(x,y,t)}{\partial x}R - \frac{\partial \vec{F}(x,y,t)}{\partial y}S + \frac{\partial \vec{F}(0,y,t)}{\partial y}S'' + \vec{F}(x,y,t)Q \tag{2}$$

where $S'' = S - S' = \mu_1 I$. Thus at equilibrium, when this exists,

$$\vec{F}_x(x,y,t)R + \vec{F}_y(x,y,t)S - \vec{F}_y(0,y,t)S'' = \vec{F}(x,y,t)Q \tag{3}$$

for $x, y \geq 0$. Finally, we have $\vec{F}(x,0) = \vec{F}(0,0)$ for $x \geq 0$, since X must be 0 whenever $Y = 0$, and the further boundary condition at infinity that $\vec{F}(\infty,\infty) = \vec{\pi}$.

We proceed in exactly the same way as for the single fluid queue, but in two dimensions (corresponding to the fluid levels in the two queues) and taking into account the boundary equation at $x = 0$.

We use the following notation:

$$\begin{aligned}
\vec{F}^{*\cdot}(\theta,y) &= \int_0^\infty e^{-\theta x}\vec{F}(x,y)\mathrm{d}x \\
\vec{F}^{\cdot *}(x,\phi) &= \int_0^\infty e^{-\phi y}\vec{F}(x,y)\mathrm{d}y \\
\vec{f}^{*\cdot}(\theta,y) &= \int_0^\infty e^{-\theta x}\frac{\partial \vec{F}}{\partial x}\mathrm{d}x \\
\vec{f}^{\cdot *}(x,\phi) &= \int_0^\infty e^{-\theta y}\frac{\partial \vec{F}}{\partial y}\mathrm{d}y
\end{aligned}$$

i.e. the superscripted 'dot' indicates a *distribution* function of a random variable and the asterisk denotes the Laplace transform of a density of a random variable. Thus, in particular, $\vec{f}^{*\cdot}(0,y) = F_Y(y)$ and $\vec{f}^{\cdot *}(x,0) = F_X(x)$ denote the marginal distributions of the fluid levels at equilibrium.

The (Laplace transform of the) solution of the differential equations 3,1 for $\vec{f}^*(\theta,\phi)$ is given by the following.

Theorem 3 *The joint probability density function of the fluid levels in queues 1 and 2 at equilibrium has Laplace transform*

$$\vec{f}^*(\theta,\phi) = \left(\frac{\gamma_2\mu_2}{r_1(\theta - \Theta^-(\phi))(\mu_1 + r_2\Theta^+(\phi)/\phi)}\right)(b, \theta r_1 + \phi s + a)$$

where

$$\Theta^{\pm}(\phi) = \frac{-\phi s(r_1 + r_2) - br_1 - ar_2 \mp d}{2r_1 r_2}$$

are the positive and negative roots of the quadratic equation

$$r_1 r_2 \theta^2 + [\phi s(r_1 + r_2) + br_1 + ar_2]\theta + \phi s(a + b) + \phi^2 s^2 = 0$$

and where

$$d = \sqrt{[\phi s(r_1 + r_2) + br_1 + ar_2]^2 - 4r_1 r_2 \phi s(a + b + \phi s)}$$

.

The proof is detailed in Appendix B

3.2 Moments

The joint moments of the fluid levels are obtained by differentiating the Laplace transform $\vec{f^*}$ at $\theta = \phi = 0$. We define the following notation for $i, j \geq 0$:

$$\begin{aligned}
M_{ij} &= (-1)^{i+j} \frac{\partial^i}{\partial \theta^i} \frac{\partial^j}{\partial \phi^j} \vec{f^*}(\theta, \phi) \Big|_{\theta=\phi=0} \\
A(\phi) &= \sqrt{[(r_1 + r_2)s\phi + br_1 + ar_2]^2 - 4r_1 r_2 s\phi(a + b + s\phi)} \\
H(\theta, \phi) &= \theta - \Theta^{-}(\phi) \\
J(\phi) &= \mu_1 + r_2 \Theta^{+}(\phi)/\phi \\
H_{ij} &= \frac{\partial^i}{\partial \theta^i} \frac{\partial^j}{\partial \phi^j} H(\theta, \phi) \Big|_{\theta=\phi=0} \\
J_j &= \frac{\partial^j}{\partial \phi^j} J(\phi) \Big|_{\phi=0}
\end{aligned}$$

Note that M_{ij} is the joint i-j moment of the fluid levels X and Y.

Equation B.4 can now be written in the form

$$\vec{f^*}(\theta, \phi) H(\theta, \phi) J(\phi) = \frac{\mu_2 \gamma_2}{r_1} (b, a + \theta r_1 + \phi s) \tag{4}$$

and

$$\Theta^{\pm}(\phi) = \frac{-[(r_1 + r_2)s\phi + br_1 + ar_2] \mp A(\phi)}{2r_1 r_2}$$

The moments of the fluid levels X and Y then follow from the derivatives of $H(\theta, \phi)$ and $J(\phi)$, which in turn are given by those of $A(\phi)$. We obtain the following low-order derivatives of the functions H and J at $\theta = \phi = 0$.

Lemma 1

$$
\begin{aligned}
H_{00} &= \frac{\gamma_1(a+b)}{r_1} \\
H_{01} &= \frac{s}{\mu_1\gamma_1} + \frac{(r_1+r_2)s}{r_1 r_2} \\
H_{02} &= -\frac{2abs^2\lambda^2}{\mu_1^3\gamma_1^3(a+b)^3} \\
H_{10} &= 1 \\
H_{11} &= 0 \\
H_{20} &= 0 \\
J_0 &= \frac{\mu_2\gamma_2}{\gamma_1} \\
J_1 &= -\frac{r_2 abs^2\lambda^2}{\mu_1^3\gamma_1^3(a+b)^3} \\
J_2 &= \frac{2r_2 abs^3\lambda^3(br_1 - ar_2)}{\mu_1^5\gamma_1^5(a+b)^5}
\end{aligned}
$$

Proof Rather tedious algebra yields the following derivatives of $A(\phi)$ at $\phi = 0$:

$$
\begin{aligned}
A(0) &= -(br_1 + ar_2) = \mu_1\gamma_1(a+b) \\
A'(0) &= \frac{s\lambda(br_1 - ar_2)}{\mu_1\gamma_1(a+b)} \\
A''(0) &= \frac{4abr_1r_2s^2\lambda^2}{\mu_1^3\gamma_1^3(a+b)^3} \\
A'''(0) &= \frac{-12abr_1r_2s^3\lambda^3(br_1 - ar_2)}{\mu_1^5\gamma_1^5(a+b)^5}
\end{aligned}
$$

We further have, recalling that $\Theta^+(0) = 0$ and using l'Hopital's rule,

$$
\lim_{\phi\to 0} \frac{\partial}{\partial\phi}\left(\frac{\Theta^+(\phi)}{\phi}\right) = \frac{1}{2}\left.\frac{\partial^2\Theta^+}{\partial\phi^2}\right|_0 \qquad \lim_{\phi\to 0} \frac{\partial^2}{\partial\phi^2}\left(\frac{\Theta^+(\phi)}{\phi}\right) = \frac{1}{3}\left.\frac{\partial^3\Theta^+}{\partial\phi^3}\right|_0
$$

The derivatives stated in the lemma then follow. ♠

Now let $K_{ij} = \frac{\partial^i}{\partial\theta^i}\frac{\partial^j}{\partial\phi^j}H(\theta,\phi)J(\phi)\Big|_{\theta=\phi=0}$ for $i, j \geq 0$. Then we have

$$
\begin{aligned}
K_{00} &= H_{00}J_0 \\
K_{01} &= H_{01}J_0 + H_{00}J_1 \\
K_{02} &= H_{02}J_0 + 2H_{01}J_1 + H_{00}J_2 \\
K_{10} &= H_{10}J_0 \\
K_{11} &= H_{10}J_1 \\
K_{20} &= 0
\end{aligned}
$$

The lower (joint) moments of the fluid levels at equilibrium now follow in terms of the quantities K_{ij}.

Proposition 4

$$
\begin{aligned}
\overline{X} \;\equiv\; M_{10} &= \frac{(\lambda-\mu_1)b}{\mu_1\gamma_1(a+b)^2}(\mu_1, \lambda-\mu_1) \\
\overline{Y} \;\equiv\; M_{01} &= \frac{K_{01}(\lambda-\mu_1)\vec{\pi} - \mu_2\gamma_2(0, \mu_1-\mu_2)}{\mu_2\gamma_2(a+b)} \\
M_{11} &= \frac{-K_{11}\vec{\pi} + K_{01}M_{10} + K_{10}M_{01}}{K_{00}} \\
M_{20} &= \frac{2K_{10}M_{10}}{K_{00}} \\
M_{02} &= \frac{2K_{01}M_{01} - K_{02}\vec{\pi}}{K_{00}}
\end{aligned}
$$

Proof Differentiating 4 at $\theta = \phi = 0$ repeatedly yields:

$$
\begin{aligned}
\vec{\pi}K_{10} - M_{10}K_{00} &= \mu_2\gamma_2(0,1) \\
\vec{\pi}K_{01} - M_{01}K_{00} &= \mu_2\gamma_2(0, s/r_1) \\
\vec{\pi}K_{11} - M_{10}K_{01} + M_{11}K_{00} - M_{01}K_{10} &= 0 \\
-2M_{10}K_{10} + M_{20}K_{00} &= 0 \\
\vec{\pi}K_{02} - 2M_{01}K_{01} + M_{02}K_{00} &= 0
\end{aligned}
$$

The proposition then follows by straightforward rearrangement. ♠

Notice that the results for M_{10}, M_{20} match those for the single fluid queue given by equations A.3, A.4, A.5.

	Mean			Standard deviation		
μ_2	Exact	$n = 2$	$n = 3$	Exact	$n = 2$	$n = 3$
10	6.94	6.075	6.94	7.87	6.72	7.79
11	2.89	2.531	2.89	3.63	3.04	3.56
12	1.54	1.350	1.54	2.10	1.74	2.05

Table 1
Exact and approximate results for fluid level at the second queue (light load)

4 Approximation Method

In a conventional Markovian queueing network the solution for the joint queue length probability distribution is separable and can be expressed as the product of the marginal queue length distributions. In a similar vein, for an arbitrary network of fluid queues we seek an approximate compositional method for analysing each queue in isolation.

The output process from each queue in the network is itself *on/off* in nature. The *off*-periods have the same distribution as the *off*-periods of the Markov modulated process at the network's input (exponential, rate b). The *on*-periods correspond to the busy periods of the queue and so are not exponentially distributed in length. Our approach is based on using an n-state Markov chain to approximate these intermediate *on/off* processes.

4.1 First Attempt: $n = 2$

The simplest approximation is the case $n = 2$, which essentially approximates the distribution of the busy period of the feeding queue by an exponential distribution with a matching mean. Suppose that the output rate of a queue during a busy period is μ, rather than λ. For this approximation we thus seek a modified rate parameter, a', such that

$$\left(\frac{b}{a' + b}\right) \mu_1 = \left(\frac{b}{a + b}\right) \lambda$$

viz. $a' = \mu_1(b + c)/\lambda - b$. The approximate results obtained in this way are shown in Table 1($n = 2$), assuming $a = b = 5, \lambda = 18, \mu_1 = 16$, with $\mu_2 = 10, 11, 12$. and, in Table 2($n = 2$), when $\lambda = 18, \mu_1 = 10$ and μ_2 varies between 9.2 and 9.6. Note that in the second case the system is stable for $9 < \mu_2 < 10$.

For the first parameterisation the approximation is reasonably good as the system is relatively lightly loaded. When the load of the first queue is increased

μ_2	Mean			Standard deviation		
	Exact	$n=2$	$n=3$	Exact	$n=2$	$n=3$
9.2	32.40	6.48	32.40	38.34	6.62	33.11
9.4	12.15	2.43	12.15	16.30	2.54	12.68
9.6	5.40	1.08	5.40	8.01	1.15	5.75
9.8	2.03	0.41	2.03	3.27	0.44	2.20

Table 2
Exact and approximate results for fluid level at the second queue (heavy load)

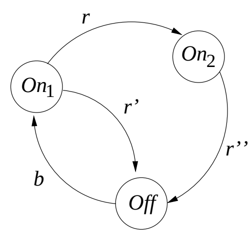

Fig. 3. Three-state approximating Markov chain

the approximation is very much worse — the busy period has a much heavier tail in this case and the exponential assumption is thus relatively poor.

4.2 Second Attempt: $n = 3$

In order to improve the approximation we now consider a three-state Markov chain, as illustrated in Figure 3. There are three unknown parameters (rates r, r' and r'') and one known parameter (b – the *off*-state parameter of the input process).

To determine the unknown parameters, we match the first three moments of the busy period distribution from the feeding queue with the first three moments of the passage time (on-period) from state Off back to itself. The former comes from Corollory 1 (Section 3.1). To compute the latter, let $H_i^*(\theta)$ be the Laplace transform of the holding time in state $i \in \{Off, On_1, On_2\}$. The Laplace transform of the *on*-period, O^*, say, is then:

$$O^*(\theta) = H_1^*(\theta)\,(q + pH_2^*(\theta))$$

By differentiating $O^*(\theta)$ m times and setting $\theta = 0$ we obtain the m^{th} moment of the *on*-period in terms of the moments of the state holding times in states 1 and 2:

$$
\begin{aligned}
O'^{*}(0) &= m_1(1) + p\, m_2(1) \\
O''^{*}(0) &= m_1(2) + 2p\, m_1(1)\, m_2(1) + p\, m_2(2) \\
O'''^{*}(0) &= m_1(3) + 3p\, m_1(2)\, m_2(1) \\
&\quad + 3m_1(1)\, m_2(2) + p\, m_2(3)
\end{aligned}
$$

where $m_i(j)$ is the j^{th} moment of the holding time in state i and $p = r/(r+r')$. Note that the holding times are exponential, so the $m_i(j)$ are straightforwardly determined.

By equating the moments we obtain three equations in three unknowns which are solved by least squares using Mathematica.

The results for the above parmeterisations are shown in Tables 1 and 2 in the column labelled "$n = 3$". The extra state in the intermediate Markov chain significantly improves the approximation achieved by the two-state chain. To two decimal places the means match precisely the exact result, even under heavy load. The standard deviations show rather more error: the approximate model consistently underestimates the exact result, with the maximum relative error varying between 1% and 33%.

Interestingly, the relative error decreases as the load increases. This is a curious aspect of the approximation that requires further investigation.

4.3 Accumulated Error

In order to evaluate the accumulated effects of the approximation error, we now consider a linear chain of ten fluid queues. The objective is to estimate the error at each point in the chain by comparing the mean and standard deviation of the fluid level with estimates from simulation.

We consider two set-ups. The first (Scenario A) comprises a relatively lightly loaded chain, with $a = 4, b = 5$, $\lambda = 18$ and fluid rate values of $11, 10.8, ..., 9.2$. In the second set-up (Scenario B), we use the same values of a, b, λ but use fluid rates of $9, 8.9, ..., 8.1$. Note that the system is stable for fluid rates greater than 8 in both cases. The queues further down the chain are progressively more loaded as their rates approach this limit.

The results for the even numbered queues (the first queue is numbered 1) are shown in Table 3. For each scenario (A or B) there are six sets of results: the mean and standard deviation from both the approximate models (i.e. $n = 2$ and $n = 3$) and the mean and standard deviation estimated by simulation. For the simulation results the half width of the 90% confidence interval is shown in parentheses on a separate row, expressed as a percentage of the point estimate above it.

		Queue Number				
Scenario	Measure	2	4	6	8	10
A	$n = 2$ Mean	0.104	0.126	0.157	0.204	0.284
	$n = 2$ S.D.	0.135	0.159	0.192	0.241	0.323
	$n = 3$ Mean	0.212	0.285	0.404	0.617	1.058
	$n = 3$ S.D.	0.277	0.361	0.501	0.803	1.423
	Sim Mean	0.213	0.285	0.404	0.619	1.060
		(0.97)	(0.33)	(0.64)	(0.71)	(0.51)
	Sim S.D.	0.367	0.493	0.688	1.048	1.765
		(0.71)	(1.40)	(1.55)	(1.61)	(1.60)
B	$n = 2$ Mean	0.198	0.260	0.372	0.635	1.950
	$n = 2$ S.D.	0.219	0.282	0.395	0.658	1.975
	$n = 3$ Mean	0.988	1.587	2.963	7.407	44.44
	$n = 3$ S.D.	1.095	1.723	3.146	8.304	50.28
	Sim Mean	0.991	1.601	2.988	7.420	44.63
		(0.42)	(0.61)	(0.74)	(0.95)	(1.50)
	Sim S.D.	1.674	2.687	4.934	11.79	63.32
		(0.60)	(0.59)	(1.56)	(1.64)	(2.37)

Table 3
Exact and approximate results for fluid level at the even-numbered queues

The accuracy of the two-state approximation degrades rapidly, again due to the poor match between busy period distribution and the exponential distribution used to approximate it. The three-state approximation holds up remarkably well, at least for the means. Even at the end of the chain, where the final node is heavily loaded the approximation lies comfortably within the 90% confidence interval obtained by simulation.

The standard deviations are again underestimated and once again there is a trend for the error to decrease as the load increases, as we observed above.

5 Summary and Conclusion

We have evaluated an approximate compositional approach to the analysis of networks of fluid queues. The method approximates the *on/off* behaviour of the intermediate links in a network using an n-state Markov chain that is parameterised by matching the first k moments of the busy period of the feeding queue with those of the passage time in the chain that is used to approximate it. The busy period moments are obtained from an exact analytical model of a single fluid queue.

The accuracy of the approximation can be computed in the context of a two-stage tandem queueing network by comparison with an exact model of the tandem queue; the latter model is obtained by generalising that for the single-queue. For larger fluid queueing networks, the accumulated error has to be evaluated by comparison with simulation.

We have shown results for two- and three-state approximating Markov chains. The two-state approximation performs poorly under high load and the accumulated error suggests that it is of little practical use in larger networks. However, the three-state model we have proposed performs remarkably well. The accuracy of the approximation when computing means was less than 1% in all the experiments we performed, including a ten-stage linear queueing network under varying load in which the error in one stage is inherited by the next. The approximation error for standard deviations was more significant, but even here, the maximum relative error was just over 50%.

We have not yet evaluated Markov chains with more elaborate structure. This is because we wanted to keep the number of moments that had to be matched to a minimum, at least to start with. It will be interesting to explore chains with additional states and, possibly, additional transitions in due course. These will require additional moments to be extracted from the chain, and the analytical tandem queue model. However, this involves only routine algebra. It is not clear how many moments can be matched exactly for a given approximating chain. For example, in our three-state system we were able to match the first two moments exactly for the parameterisations we explored.

We might expect that, for a carefully chosen structure, the error in the standard deviation (and, indeed, estimates of higher moments of the fluid level) could be reduced, when compared to the current three-state approximation.

An obvious next step is to consider more general queueing networks. In this paper we have restricted ourselves to linear chains of fluid queues with a single external MMOAP, although these can be straightforwardly generalised to tree-like networks with a single source at the root.

One such generalisation would permit multiple external input streams at any node. This would require us to determine the busy period in a queue fed with such multiple sources, although there is an exact analytical solution for this case [4]. Correspondingly, we would then have to extend the analysis for the fluid level in a single queue to handle multiple inputs. Alternatively, wherever there are multiple inputs, we could merge them to form a single MMOAP with

3^n states when there are n inputs. Then we are back to our current approach where there is one fluid input per queue. However, we would need to solve a more complex passage time problem in order to compute the moments of the busy period for the purposes of moment matching. A similar problem arises if we were to allow the outputs from two fluid queues to be merged at the inputs to another. Networks with cycles present a further interesting challenge.

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A Single fluid queue: review

Consider a single fluid queue, comprising a server that outputs fluid, a reservoir where input fluid is stored and a Markov modulated input (or arrival) stream. Suppose that:

- there are n states, in the continuous time Markov chain, which has generator matrix $Q = (q_{ij} \mid 1 \le i, j \le n)$ and equilibrium probabilities $\vec{\pi}$ (so that $\pi Q = \vec{0}$ and $\pi \vec{e}^{\mathrm{T}} = 1$, where $e = (1, 1, \ldots, 1)$);
- the arrival rate in state i is the constant λ_i volume-units of fluid per unit time;
- the rate at which the server outputs fluid when its reservoir is non-empty is μ volume-units of fluid per unit time;
- the diagonal (net input) *rate matrix* $R = \mathrm{diag}(r_1, \ldots, r_n)$, where $r_i = \lambda_i - \mu$ for $1 \le i \le n$, and the rate vector $\vec{r} = (r_1, \ldots, r_n)$.

The following, rather unrigorous, argument leads to a differential equation for the equilibrium fluid level probability distribution. The intuitive condition for equilibrium is that the average net input rate of fluid is negative above some fluid level; i.e. here, where rates are constant, it is $\vec{\pi}.\vec{r} < 0$. Let the state and the fluid level in the reservoir at time t be denoted by N_t and X_t respectively and define $\vec{F}(x,t) = (F_1(x,t), \ldots, F_n(x,t))$, where

$$F_i(x,t) = P(N_t = i, X_t \le x)$$

Now consider the infinitesimal interval $(t, t+h]$ for some small h. Then we have, to first order in h,

$$F_i(x, t+h) = (1 + q_{ii}h)F_i(x - r_i h, t) + \sum_{j \ne i} F_j(x,t) q_{ji} h + o(h)$$

Thus we have:

$$\begin{aligned} \frac{F_i(x,t+h) - F_i(x,t)}{h} &= F_i(x,t)q_{ii} - r_i \frac{\partial F_i(x,t)}{\partial x} + \sum_{j \ne i} F_j(x,t) q_{ji} + O(h) \\ &= -r_i \frac{\partial F_i(x,t)}{\partial x} + \sum_{j=1}^{n} F_j(x,t) q_{ji} + O(h) \end{aligned}$$

so that in the limit $h \to 0$

$$\frac{\partial \vec{F}}{\partial t} = -\frac{\partial \vec{F}}{\partial x} R + \vec{F} Q$$

and, at equilibrium when this exists,

$$\vec{F}_x R = \vec{F} Q$$

where we use the subscript x to denote a partial derivative with respect to x. In addition, we have the boundary conditions that $F_i(0) = 0$ if $r_i > 0$, which reflects the fact that the reservoir cannot be empty when there is a positive net input.

We solve this equation, under the further boundary condition at infinity that $\vec{F}(\infty) = \vec{\pi}$, by taking Laplace transforms. First, the differential equation for the single queue is multiplied by the appropriate exponential factor and integrated between the limits 0^+ and ∞. We have to be careful about the lower limit since the derivative $\vec{F}_x$ is unbounded at $x = 0$ since $F_2(0) \neq 0$.[1] The integral's lower limit is 0^+ because the differential equation holds only for $x > 0$.

Let the vector density function (assuming it exists so that the differential equation is meaningful) $\vec{f}(x) = \frac{\partial \vec{F}}{\partial x}$ and denote the Laplace transform operator by an asterisk, so that for example

$$\vec{F}^*(\theta) = \int_{0^-}^{\infty} e^{-\theta x} \vec{F}(x) \mathrm{d}x$$

Note that the lower limit is usually written just 0, but we have written 0^- to emphasise the above referred discontinuity in F_2 at the origin. Multiplying the differential equation throughout by $e^{-\theta x}$ and integrating from 0^+ to ∞, we obtain

$$\theta \vec{F}^*(\theta) R - \vec{F}(0) R = \vec{F}^*(\theta) Q \tag{A.1}$$

(Notice that the first integral in the above equation has lower limit 0^+. However, since $\vec{F}(x)$ is bounded at $x = 0$, the integral with lower limit replaced by 0^- has the same value.) Using the well known property (again easily established by integrating by parts) that $g^*(\theta) = \theta G^*(\theta)$ for any differentiable function G, we have:

$$\vec{f}^*(\theta) R - \vec{F}(0) R = \vec{F}^*(\theta) Q$$

Remarks

- Using the property of Laplace transforms that $g^*(\infty) = G(0)$ we find that $\vec{F}^*(\infty) Q = 0$, as expected since $\vec{F}^*(\infty) = 0$.
- Multiplying by θ and setting θ to 0, we find $\vec{f}^*(0) Q = 0$. Since in general $g^*(0) = G(\infty)$, this implies that $\vec{F}(\infty) Q = 0$, confirming that $\vec{F}(\infty) = \vec{\pi}$.

Equation A.1 can be rewritten in the form $\vec{F}^*(\theta)(\theta R - Q) = \vec{F}(0) R$ so that

$$\vec{F}^* = \vec{F}(0) R (\theta R - Q)^{-1}$$

Noting that the inverse matrix exists for $\theta > 0$, we now find

$$\vec{F}^*(\theta) = \frac{\vec{F}(0) R}{\theta(r_1 r_2 \theta + r_1 b + r_2 a)} \begin{pmatrix} r_2 \theta + b & a \\ b & r_1 \theta + a \end{pmatrix}$$

[1] In fact $\frac{\partial F_2(0)}{\partial x} = F_2(0) \delta(x)$, a Dirac delta-function.

Multiplying by θ and noting that $\vec{F}(0) = (0, F_2(0))$, we get

$$\vec{f}^*(\theta) = \frac{r_2 F_2(0)}{r_1 r_2 \theta + r_1 b + r_2 a}(b, r_1\theta + a) \tag{A.2}$$

The equilibrium condition requires $\pi_1 r_1 + \pi_2 r_2 < 0$, i.e. $r_1 b + r_2 a < 0$ or $\lambda b < \mu_1(a + b)$. Thus, since $r_2 < 0, r_1 > 0$ both the numerator and the denominator are negative for all $\theta \geq 0$.

At $\theta = 0$,

$$\vec{\pi} = \frac{r_2 F_2(0)}{r_1 b + r_2 a}(b, a)$$

But $\vec{\pi} = \frac{(b,a)}{a+b}$ and so we require $r_2(a + b)F_2(0) = r_1 b + r_2 a$, i.e.

$$F_2(0) = 1 + \frac{(r_1 - r_2)b}{r_2(a + b)} = 1 + \pi_1 \frac{(r_1 - r_2)}{r_2}$$

Substituting for the constants r_1, r_2 we find

$$F_2(0) = 1 - \pi_1 \lambda/\mu_1$$

Since $\pi_1\lambda$ is the average arrival rate of fluid and the service rate is the constant μ_1 when the reservoir is non-empty, in a steady-state we must have $\pi_1\lambda = U_1\mu_1$ where $U_1 = 1 - F_1(0) - F_2(0)$ is the utilisation of node 1. Since $F_1(0) = 0$ this equation is consistent with our result.

Finally, we can write the required solution as a Laplace transform:

$$\vec{f}^*(\theta) = (0, F_2(0)) + \frac{\pi_1\alpha}{\theta + \alpha}(1, (\lambda - \mu_1)/\mu_1) \tag{A.3}$$

where $\alpha = \frac{r_1 b + r_2 a}{r_1 r_2}$ so that $F_2(0) = \frac{\alpha r_1}{a+b}$. This is easily inverted by inspection to give

$$\vec{f}(x) = (0, F_2(0))\delta(x) + (1, (\lambda - \mu_1)/\mu_1)\pi_1\alpha e^{-\alpha x} \tag{A.4}$$

or

$$\vec{F}(x) = (0, F_2(0)) + (1, (\lambda - \mu_1)/\mu_1)\pi_1[1 - e^{-\alpha x}] \tag{A.5}$$

B Tandem pair of fluid queues

Proposition 5 *The joint probability density function of the fluid levels in queues 1 and 2 at equilibrium has Laplace transform $\vec{f}^*(\theta, \phi)$ given by the equation*

$$\vec{f}^*(\theta, \phi)(\theta R + \phi S - Q) = \phi \vec{f}^{*\cdot}(\theta, 0)S + \vec{f}^{\cdot *}(0, \phi)(\theta R + \phi S'') - \phi \vec{F}(0, 0)S''$$

and boundary conditions

$$\vec{F}(x,0) = \vec{F}(0,0) \quad \textit{for} \quad x \geq 0 \qquad \vec{F}(\infty,\infty) = \vec{\pi}$$

Proof Multiplying the differential equation 3 throughout by $e^{-\theta x}$ and integrating from 0^+ to ∞ w.r.t. x, we obtain

$$\theta \vec{F}^{*\cdot}(\theta,y)R + \vec{F}_y^{*\cdot}(\theta,y)S - \vec{F}^{*\cdot}(\theta,y)Q = \vec{F}(0,y)R + \vec{F}_y(0,y)S''/\theta$$

Multiplying throughout by $e^{-\phi y}$ and integrating from 0^+ to ∞ w.r.t. y now yields:

$$\vec{F}^{*}(\theta,\phi)(\theta R + \phi S - Q) = \vec{F}^{*\cdot}(\theta,0)S + \vec{F}^{\cdot *}(0,\phi)(R + \phi S''/\theta) - \vec{F}(0,0)S''/\theta$$

where a sole asterisk denotes a double Laplace transform. Multiplying by $\theta\phi$ now gives:

$$\begin{aligned} \vec{f}^{*}(\theta,\phi)(\theta R + \phi S - Q) = {} & \phi \vec{f}^{*\cdot}(\theta,0)S \\ & + \vec{f}^{\cdot *}(0,\phi)(\theta R + \phi S'') \\ & - \phi \vec{F}(0,0)S'' \end{aligned} \tag{B.1}$$

as required and the boundary conditions have already been obtained. ♠

From this result we can derive the marginal probability that there is no fluid at either node.

Corollary 6 *The marginal equilibrium probabilities that the fluid levels are zero at nodes 1 and 2 are respectively:* $\vec{F}_X(0) = (0,\gamma_1)$ *and* $\vec{F}_Y(0) = (0,\gamma_2)$ *where* $\gamma_1 = 1 - \pi_1\lambda/\mu_1$ *and* $\gamma_2 = 1 - \pi_1\lambda/\mu_2$*. Further,* $\vec{F}(0,0) = \vec{F}_Y(0)$*.*

Proof When $\phi = 0$ we have $\vec{f}^{*}(\theta,0)(\theta R - Q) = \vec{f}^{\cdot *}(0,0)\theta R$ or

$$\vec{f}_X^{*}(\theta)\theta^{-1}(\theta R - Q) = \vec{F}_X^{*}(\theta)(\theta R - Q) = \vec{F}_X(0)R$$

This describes exactly the single fluid queue and so, as in Appendix A, $\vec{F}_X(0) = (0, 1 - \pi_1\lambda/\mu_1) = (0,\gamma_1)$.

Next, let $\theta = 0$. This gives

$$\begin{aligned}\vec{f}^*(0,\phi)(\phi S - Q) &= \phi(\vec{f}^{*\cdot}(0,0)S + \vec{f}^{\cdot *}(0,\phi)S'' - \vec{F}(0,0)S'') \\ &= \phi(\vec{F}_Y(0)(S - S'') + \vec{f}^{\cdot *}(0,\phi)S'') \\ &= \phi(\vec{F}_Y(0)S' + \vec{f}^{\cdot *}(0,\phi)S'')\end{aligned}$$

since X must be 0 whenever $Y = 0$, and so $\vec{F}(0,0) = \vec{F}_Y(0) = (0, F_{Y2}(0))$ (proving the last equation of the corollary). Thus, post- multiplying by the matrix $(\phi S - Q)^{-1}$, as in Appendix A, we obtain

$$\vec{f}^*(0,\phi) = \frac{\vec{F}_Y(0)S' + \vec{f}^{\cdot *}(0,\phi)S''}{s(\phi s + a + b)} \begin{pmatrix} \phi s + b & a \\ b & \phi s + a \end{pmatrix} \tag{B.2}$$

where (recall) $s = \mu_1 - \mu_2$. Thus, at $\phi = 0$,

$$\vec{f}^*(0,0) = \vec{\pi} = \frac{\beta(b,a)}{s(a+b)} = \frac{\beta}{s}\vec{\pi}$$

where $(0,\beta) = \vec{F}_Y(0)S' + \vec{F}_X(0)S'' = \mu_1\vec{F}_X(0) - \mu_2\vec{F}_Y(0)$, so $\beta = \mu_1 - \pi_1\lambda - \mu_2 F_{Y2}(0)$. To ensure the boundary condition at equilibrium, we must have $\beta = s$, and so $\mu_1 - \pi_1\lambda - \mu_2 F_{Y2}(0) = \mu_1 - \mu_2$ giving

$$F_{Y2}(0) = 1 - \frac{\pi_1\lambda}{\mu_2} \qquad \spadesuit$$

This result could have been derived by steady-state arguments, the average arrival rate at both nodes being $\pi_1\lambda + \pi_2.0$, giving utilisations $\pi_1\lambda/\mu_1$ and $\pi_1\lambda/\mu_2$.

Using the single queue solution, equation A.2 of Appendix A, we have

$$\vec{f}^*(\theta,0) = \frac{r_2\gamma_1}{r_1 r_2\theta + r_1 b + r_2 a}(b, r_1\theta + a)$$

However, we cannot simply obtain the general solution for $\vec{f}^*(\theta,\phi)$, nor indeed for $\vec{f}^*(0,\phi)$ since we do not know $\vec{f}^{\cdot *}(0,\phi)$ in equation B.2.

B.1 Computation of $\vec{f}^{*\cdot}(\theta,0)$ and $\vec{f}^{\cdot *}(0,\phi)$

In this simple network, we have already noted that $\vec{F}(x,0) = \vec{F}(0,0) \ \ \forall x \geq 0$ since the fluid level is zero at the second queue only if it is also zero at the first queue. Hence $\vec{f}^{*\cdot}(\theta,0) = \vec{F}(0,0) = (0,\gamma_2)$. The calculation of $\vec{f}^{\cdot *}(0,\phi)$ is more complex.

Lemma 2

$$f_2^{\cdot *}(0,\phi) = \frac{\phi\gamma_2(\mu_1 - s)}{\Theta^+(\phi) r_2 + \phi\mu_1}$$

where

$$\Theta^+(\phi) = \frac{-\phi s(r_1 + r_2) - br_1 - ar_2 - d}{2r_1 r_2}$$

where $d = \sqrt{[\phi s(r_1 + r_2) + br_1 + ar_2]^2 - 4r_1 r_2 \phi s(a + b + \phi s)}$

Proof We consider equation B.1 and the singularities of the matrix $M(\theta, \phi) = \theta R + \phi S - Q$. When M is singular, for certain values of the pair (θ, ϕ), there is a right-eigenvector $\vec{e}(\theta, \phi)$ such that $M\vec{e} = \vec{0}$ and hence

$$\vec{v}.\vec{e} = 0$$

where $\vec{v} = \phi \vec{f}^{*\cdot}(\theta, 0)S + \vec{f}^{\cdot *}(0, \phi)(\theta R + \phi S'') - \phi \vec{F}(0, 0)S''$. Now, M is singular when

$$\begin{vmatrix} \theta r_1 + \phi s + a & a \\ b & \theta r_2 + \phi s + b \end{vmatrix} = 0$$

This simplifies to the quadratic in θ

$$\Delta(\theta, \phi) \equiv r_1 r_2 \theta^2 + [\phi s(r_1 + r_2) + br_1 + ar_2]\theta + \phi s(a + b) + \phi^2 s^2 = 0$$

Since $r_1 > 0, r_2 < 0, s = \mu_1 - \mu_2 > 0$, the coefficient of θ^2 is negative and the constant term is positive for all $\phi > 0$. Consequently, $\Delta(-\infty, \phi) < 0, \Delta(0, \phi) > 0$ and $\Delta(+\infty, \phi) < 0$. Hence there is exactly one positive root and one negative root. When $\phi = 0$, the equation is

$$\theta(r_1 r_2 \theta + br_1 + ar_2) = 0$$

which has roots 0 and $-(br_1 + ar_2)/r_1 r_2$ which is negative. The positive root is therefore (remembering that the denominator is negative)

$$\Theta^+(\phi) = \frac{-\phi s(r_1 + r_2) - br_1 - ar_2 - d}{2r_1 r_2} \tag{B.3}$$

for all $\phi \geq 0$. d is as defined above. The corresponding eigenvector is then $\vec{e} = (e_1(\phi), e_2(\phi))$ where

$$\frac{e_1}{e_2} = -\frac{a}{r_1 \Theta^+(\phi) + s\phi + a}$$

Since the denominator is never zero (nor the numerator for that matter), we can choose $e_2 = 1$ and

$$\vec{e} = \left(-\frac{a}{r_1\Theta^+(\phi) + s\phi + a}, 1 \right)$$

Now, $\vec{f}^{*\cdot}(\theta, 0) = (0, \gamma_2)$ and $\vec{f}^{\cdot *}(0, \phi) = (0, f_2^{\cdot *}(0, \phi)$. Hence,

$$\vec{v}.\vec{e} = v_2 = \phi s \gamma_2 + f_2^{\cdot *}(0, \phi)(\Theta^+(\phi) r_2 + \phi\mu_1) - \phi\gamma_2\mu_1 = 0$$

and so

$$f_2^{\cdot *}(0, \phi) = \frac{\phi\gamma_2(\mu_1 - s)}{\Theta^+(\phi) r_2 + \phi\mu_1} \qquad \spadesuit$$

As a check, notice that

$$\lim_{\phi \to 0} \frac{\Theta^+(\phi)}{\phi} = -\frac{s(a+b)}{br_1 + ar_2}$$

and that as $\phi \to 0$, $f_2^{\cdot *}(0, \phi) \to \gamma_1$ as required.

B.2 Joint Laplace transform

The joint Laplace transform is given by the following theorem, repeated from Section 3.1.

Theorem 7 *The joint probability density function of the fluid levels in queues 1 and 2 at equilibrium has Laplace transform*

$$\vec{f}^*(\theta, \phi) = \left(\frac{\gamma_2\mu_2}{r_1(\theta - \Theta^-(\phi))(\mu_1 + r_2\Theta^+(\phi)/\phi)} \right) (b, \theta r_1 + \phi s + a)$$

where

$$\Theta^\pm(\phi) = \frac{-\phi s(r_1 + r_2) - br_1 - ar_2 \mp d}{2r_1r_2}$$

are the positive and negative roots of the quadratic equation

$$r_1r_2\theta^2 + [\phi s(r_1 + r_2) + br_1 + ar_2]\theta + \phi s(a+b) + \phi^2 s^2 = 0$$

and where d is as above.

Proof From equation B.1

$$\begin{aligned}
&\vec{f}^*(\theta, \phi) \\
&= \phi\gamma_2 \left(0, s + \frac{(\mu_1 - s)(\theta r_2 + \phi\mu_1)}{\Theta^+(\phi) r_2 + \phi\mu_1} - \mu_1\right) [\theta R + \phi S - Q]^{-1} \\
&= \left(\frac{\gamma_2 r_2 (\mu_1 - s)(\theta - \Theta^+(\phi))}{\Delta(\theta, \phi)(\mu_1 + r_2\Theta^+(\phi)/\phi)}\right)(b, \theta r_1 + \phi s + a) \\
&= \left(\frac{\gamma_2 r_2 (\mu_1 - s)}{r_1 r_2 (\theta - \Theta^-(\phi))(\mu_1 + r_2\Theta^+(\phi)/\phi)}\right)(b, \theta r_1 + \phi s + a) \\
&= \left(\frac{\gamma_2 \mu_2}{r_1 (\theta - \Theta^-(\phi))(\mu_1 + r_2\Theta^+(\phi)/\phi)}\right)(b, \theta r_1 + \phi s + a)
\end{aligned}$$

for all $\theta, \phi \geq 0$ ♠.