Refinement is complete for implementations

Michael Huth

Department of Computing, Imperial College London,
London, SW7 2AZ, United Kingdom

Abstract. Modal transition systems specify sets of implementations, their refining labelled transition systems, through Larsen & Thomsen’s co-inductive notion of refinement. We demonstrate that refinement precisely captures the identification of a modal transition system with its set of implementations: refinement is reverse containment of sets of implementations. This result extends to models that combine state and event observables and is drawn from a SFP-domain whose elements are equivalence classes of modal transition systems under refinement, and abstraction-based finite-model properties proved in this paper. As a corollary, validity checking is model checking for Hennessy-Milner formulas that characterize modal transition systems with bounded computation paths. We finally sketch how techniques developed in this paper can be used to detect inconsistencies between multiple modal transition systems and, if consistent, to verify properties of all common implementations.

Keywords: Model checking, refinement, modal transition systems, implementation relation.

1. Introduction

Formal modelling of computing systems and analysis of such models have always been key techniques of the formal-methods communities. We mention requirement elicitation, program analysis, many software- validation techniques, and protocol verification as beneficiaries of modelling and analysis of models. One such technique, model-checking [CE81, QS81], has been extremely successful as a formal method. In model-checking, a model $M$ captures relevant aspects of a computer artifact, a formula $\phi$ denotes static or dynamic goals that the artifact should meet, and $M \models \phi$ holds iff goal $\phi$ is met by model $M$. Using conjunction we may assume in this discussion that only one goal $\phi$ needs to be checked. Since $M$ is an abstraction of some concrete artifact $C$, i.e. $C$ is a refinement of $M$, we desire that goals verified on the abstraction hold in the concrete artifact: for all $\phi$, $M \models \phi$ should imply $C \models \phi$. If the judgment $M \models \phi$ is undecidable or has too high complexity, we want to find an abstraction $A$ of $M$ such that $A \models \phi$ implies $M \models \phi$ for all $\phi$. In either case we are concerned with defining an efficiently decidable notion of refinement, whose relational inverse is abstraction, such that certifications of goals are preserved under refinement. Once such a framework for

Correspondence and offprint requests to: Michael Huth, Department of Computing, Imperial College London, London, SW7 2AZ, United Kingdom, e-mail: M.Huth@doc.imperial.ac.uk
abstraction-based model checking is in place, we may also use it to check loose specifications $M$ that model a potentially infinite set of concrete artifacts $C$.

The vast majority of model-checking frameworks utilize labelled transition systems as models, perhaps allowing for state-based or quantitative variations thereof. Properties $\phi$ are then typically expressed in a temporal logic such as computation tree logic [CE81, QS81] or linear-time temporal logic [GPVW95]. The tremendous success of this approach to formal verification is tarnished by two related short-comings:

1. models may be specifications and the under-specification of certain aspects may not be expressible; or
2. models may be abstractions of models or artifacts but sound reasoning for goals that mix path quantifiers is required; or the transfer of verifications and refutations of goals from an abstract to the abstracted model is desired — none of these are catered for by the standard approach of abstraction for model checking in [CGL94].

Example 1.1. To illustrate the first point, consider the specification of a global assembly cache within the Microsoft .NET framework [EJS03]. Such a cache $C$ consists of a set of components $c$. Each component $c \in c$ offers a set of services $export$ to other components and requires a set of services $import$ from other components. A service could be a type, a field, a method etc. A component may have a service called main that would start the execution of software. But a component does not necessarily have such a service, e.g. the component could provide “plug-ins” only.

This optional specification of structure is not possible in labelled transition systems or their state-based versions. Similarly, labelled transition systems cannot specify that “Event $\alpha$ could possibly lead from state $s$ to state $s'$ but is not guaranteed to have that ability.” Either the triple $(s, \alpha, s')$ is in the transition relation of the labelled transition system in question, and then such a behavior is always possible, or it isn’t, and then such a behavior is always impossible.

To appreciate the second point, let $M'$ be an abstraction of $M$ and let us say that goal $\phi$ is verified in $M'$ iff $M' \models \phi$ holds; otherwise, $\phi$ is refuted in $M'$. It is then elementary to see that the sound transfer of verifications and verifications from $M'$ to $M$ requires that for all goals $\phi$ we have $(M \models \phi \iff M' \models \phi)$: Suppose that $M' \models \phi$ holds, then $\phi$ is verified at $M'$ and so the sound transfer of verifications yields that $M \models \phi$ holds; conversely if $M' \not\models \phi$ holds, then $\phi$ is refuted at $M'$ and the sound transfer of verifications renders $M \not\models \phi$. If goals subsume Hennessy-Milner logic [HM85], this forces $M$ and $M'$ to be bisimilar which, as noted by Larsen & Thomsen in [LT88], is too restrictive for aggressive abstraction techniques of state-space compression needed in model checking.

One can show that this limitation is closely related to the problems encountered with the mix of path quantifiers [HJS01]. A labelled transition system $M$ may have another labelled transition system $M_{safe}$ as a safe simulation that can match all transitions from $M$ co-inductively. Then every trace of events observable for $M$ is also a possible trace in $M_{safe}$. So if we restrict reasoning to universal path properties such as “on all paths, if an alarm is triggered, the monitor will inevitable receive this alarm," then verifications on $M_{safe}$ are sound for $M$ [CC77]. Dually, one may develop a notion of live simulation for which verifications of existential path properties such as “there is always a way to reach a stable state again” is sound. Mixing universal and existential path properties, e.g. “at all reachable states there is a path to some stable state,” naturally leads to the formulation of 3-valued models and their checks [CC00].

Such a mix may even occur if it is not discernible from the structure of the formula $\phi$ as can be seen by the problem of checking reachability in linear-time temporal logic or computation-tree logic, a universal path quantifier, in the presence of simple fairness constraints, an existential path quantifier. This mix of quantifiers and the need for abstraction-based model checking are even harder to escape in modelling applications that require path constraints not expressible in temporal logics. We mention the extension of model checking to hybrid logics in the context of modelling mobility and agents in distributed systems [FrR03], where existential constraints are needed to express the unique location of agents and universal constraints are needed to specify safety properties.

By now it is well understood that both of the two short-comings above can be addressed within a 3-valued model-checking framework [Lar89, Dam96, BG99, BG00]. In 3-valued models, an atomic observable, e.g. “in state $s$ an event $\alpha$ can lead to state $s'$,” can not only be “true” or “false” but may be under-specified in that it could be optional, “true or false,” empowering specifiers and implementors alike with more degrees of freedom. In Example 1.1 the signature declaration “a component may have a main service” is optional in this sense. In identifying scalars with singletons we may write $tt = \{tt\}$, $ff = \{ff\}$, and $\perp = \{tt, ff\}$ for the values of these observables, respectively. Note that we can determine the value of observables by asking
and answering the questions “Is the observed value $t$? And if not, is it $\bot$?” as two “no” replies determine that the value of the observable is $f$. Therefore any 3-valued model $M$ can be presented as a pair of two two-valued models $(M^a, M^c)$, where $t$ and $f$ are interpreted as $t$ and $f$ in $M^a$ and in $M^c$, respectively; and $\bot$ is interpreted as $f$ in $M^a$ and as $t$ in $M^c$; see Figure 1.

1.1. Completeness of refinement for models of propositional logic

For sake of illustration and in order to gently stage the development of our technical material, we first sketch a 3-valued model-checking framework for propositional logic. The grammar for formulas is

$$\phi ::= p \mid \neg \phi \mid \phi \land \phi$$

where $p$ ranges over a countable set of propositional variables $\text{Var}$. We write $\phi \lor \psi$ for $\neg (\neg \phi \land \neg \psi)$. A model $M$ is a total function $M : \text{Var} \rightarrow \{t, f, \bot\}$. Model $M$ is a standard 2-valued model if $\bot$ is not in the range of $M$, whereas a value $M(p) = \bot$ expresses that model $M$ is uncertain about the truth value of $p$. This uncertainty is best expressed non-deterministically by identifying $t$ with $\{t\}$, $f$ with $\{f\}$, and $\bot$ with $\{t, f\}$. This identification immediately gives us the compositional weak semantics, Kleene’s strong interpretation of 3-valued propositional logic [Kle52], by applying logical connectives point-wise and collecting results as a set. For example, $\bot \land t = \bot$ since

$$\{t, f\} \land \{t\} = \{t \land t, f \land t\} = \{t, f\}. \quad (2)$$

Similarly, we obtain $\neg \bot = \bot$, $\bot \lor \bot = \bot$ etc.

**Example 1.2.** For a model $M$ with $M(p) = \bot$, the tautology $p \lor \neg p$ evaluates to $\bot \lor \neg \bot = \bot$. This loss of precision suggests that there is a more precise, strong semantics for 3-valued propositional semantics.

Given two models $M, M' : \text{Var} \rightarrow \{t, f, \bot\}$ we say that $M$ refines, is abstracted by, $M'$ iff for all $p \in \text{Var}$ we have $M'(p) \leq M(p)$ where $\bot$ is the least element and $t$ and $f$ are maximal, incomparable elements with respect to $\leq$. One can quickly see that a model $M'$ has no refinements other than itself iff $M'$ contains no uncertainty, i.e., iff $\bot$ is not in the range of $M'$. Otherwise, if $M'$ maps exactly $k < \infty$ many variables to $\bot$, then $M'$ has $2^k$ many maximal refinements and one could define a strong semantics by appealing to those $2^k$ models [Bla80]. If we call maximal refinements “implementations” we can show that implementations completely determine refinement.

**Proposition 1.1.** For any models $M, M' : \text{Var} \rightarrow \{t, f, \bot\}$ we have that $M$ refines $M'$ iff all implementations of $M$ are also implementations of $M'$.

**Proof.** Since refinement is a transitive relation the “only if” part is shown. Conversely let all implementations of $M$ be implementations of $M'$. We need to show that $M$ refines $M'$. For $p \in \text{Var}$ it suffices to show $M'(p) \leq M(p)$. Proof by contradiction: If $M'(p) \ngeq M(p)$, then $M'(p) \neq \bot$. Without loss of generality $M'(p) = t$. If $M(p) = t$ we are done. Otherwise, $M(p) = f$. Let $M'^{it} : \text{Var} \rightarrow \{t, f\}$ be defined by $M'^{it}(q) = f$ if $M(q) = \bot$ and $M'^{it}(q) = M(q)$ otherwise. Then $M'^{it}$ is an implementation of $M$ and therefore an implementation of $M'$ by assumption. So $tt = M'(p) \leq M'^{it}(p) = ff$ is a contradiction. \qed

The proof above is very simple and would hardly be worth mentioning. But this paper sets out to generalize this fact from propositional models to behavioral models in which we have finitely many events and not only one but possibly infinitely many states.

1.2. Refinement for behavioral models

Three-valued labelled transition systems can be represented as Larsen & Thomsen’s modal transition systems [LT88]. Figure 2 depicts two modal transition systems. A modal transition system $M = (M^a, M^c)$ is such that, for each mode $m \in \{a, c\}$, $M^m$ is a labelled transition system $(\Sigma, R^m)$ where $\Sigma$ is a set of states and $R^m \subseteq \Sigma \times \text{Act} \times \Sigma$ is a transition relation over a set of events $\text{Act}$; furthermore, $R^a$ is contained in $R^c$. If we think of a modal transition system as a specification, we must ask what are its implementations. Intuitively, a
Fig. 1. On the right: a 3-valued transition system with two states and a set $\text{Act} = \{\alpha\}$ of possible events. On the left: the corresponding interpretation which shows only may-transitions $R^c \setminus R^u$ (with value $\perp$ on the right, dashed lines in figures) and solid must-transitions $R^u$ (with value $\sharp$, solid lines in figures); "transitions" with value $ff$ are omitted.

![Diagram](image)

Fig. 2. Two modal transition systems with distinguished filled, initial states. The one to the right refines the one to the left.

refinement $M'$ of $M$ resolves some may-transitions ($\perp$ values, elements of $R^c \setminus R^u$, dashed lines in figures) to must-transitions ($tt$ values, elements of $R^u$, solid lines in figures) or to absent transitions ($ff$ values, elements of the complement of $R^c$) and honors all $tt$ and $ff$ values of $M$. The resolution of $\perp$ to $tt$ constitutes a choice to implement, whereas a resolution of $\perp$ to $ff$ reflects a choice not to implement optional behavior. This naive intuition is being formalized in Larsen’s co-inductive definition of refinement $M \prec M'$ [LT88]; $M'$ refines/is abstracted by $M$, generalizing bisimulation to the 3-valued setting; see Figure 2. Definition 2.2 re-states the notion of refinement formally. An implementation $I$ of $M$ should accordingly resolve all may-transitions to must-transitions or absent transitions, forcing $I^u = I^c$, and honor all $tt$ and $ff$ values of $M$, as can be seen in Figure 3. One could summarize this discussion of refinement into a slogan: “Must stays Must, May may be Must or Absent, and only Must and May may cause a Must.” In this way, we recognize implementations as refining labelled transition systems and may define the class of implementations of $M$ as

$$I[M] = \{I \text{ modal transition system} \mid I^u \text{ equals } I^c \land M \prec I\}.$$  

(3)

Since refinement $\prec$ is transitive [Lar89], it is immediate that $M \prec M'$ implies $I[M'] \subseteq I[M]$; all implementations of a refinement are implementations of what is being refined. So refinement of modal transition systems is sound if modal transition systems are interpreted as their respective classes of implementations since subsequent refinements cannot introduce new implementations.

In this paper we ask, and answer affirmatively, whether refinement is complete for this interpretation:

“For all modal transition systems $M$ and $M'$:

If all implementations of $M'$ are implementations of $M$, does $M'$ refine $M'$?”

(4)

Therefore, refinement does not lose any precision if interpreted as reverse inclusion of sets of implementations, the scenario on the left of Figure 4 is the norm whereas the scenario on the right is impossible. Answering the question in (4) is non-trivial for two reasons. First, refinement can be captured as a winning strategy in a two-person game and our proofs depend on the ability to dynamically synthesize winning strategies out of implementations. Second, such a synthesis of winning strategies from implementations may be difficult to

![Diagram](image)

Fig. 3. Two modal transition systems with filled initial states. The one to the right implements the one to the left.
obtain for infinite-state or cyclic models and we seem to require a tool that allows us to restrict attention to certain finite-state models. We address the latter problem by expressing the model-checking framework for modal transition systems in a mathematical universe \( \mathcal{D} \) that plays a double role \([\text{HJS}04]\) as a SFP-domain \([\text{AJS94}]\) and, up to refinement equivalence, as a modal transition system. A side effect of our affirmative answer is that \( T[M'] \subseteq T[M] \) can be verified or refuted by always winning a game that involves \( M \) and \( M' \) only. We develop more consequences of this completeness result toward the end of this paper.

### 1.3. Weak and strong semantics of Hennessy-Milner logic

For the technical development of this paper we need to define \( M \models^a \phi \), the weak semantics of Larsen \([\text{Lar89}]\) denoted by \( \models \) in loc. cit., and defined over formulas \( \phi \) of Hennessy-Milner logic \([\text{HM85}]\)

\[
\phi ::= a | \neg \phi | \langle \alpha \rangle \phi | \phi \land \phi
\]

where \( a \) ranges over a finite set of events \( \text{Act} \). Larsen’s weak semantics for a pointed modal transition system \( (N, i) = ((\Sigma, R^c) \subseteq \Sigma \times \text{Act} \times \Sigma, R^c) \subseteq \Sigma \times \text{Act} \times \Sigma), i) \), a modal transition system \( N \) with a designated initial state \( i \), is given by two judgments

\[
\begin{align*}
(N, i) &\models^a \phi \quad \text{\( \phi \) is asserted at state \( i \) in \( N \)} \\
(N, i) &\models^c \phi \quad \text{\( \phi \) may be consistent at state \( i \) in \( N \)}
\end{align*}
\]

which are defined by structural induction on \( \phi \):

\[
\begin{align*}
(N, i) &\models^a \top \\
(N, i) &\models^c \bot \\
(N, i) &\models^a \neg \phi \iff (N, i) \not\models^c \phi \\
(N, i) &\models^c \neg \phi \iff (N, i) \not\models^a \phi \\
(N, i) &\models^m \langle \alpha \rangle \phi \text{ (for some } (i, i') \in R^m, (N, i') \models^m \phi \text{ )} \\
(N, i) &\models^m \phi \land \psi \iff ((N, i) \models^m \phi \text{ and } (N, i) \models^m \psi)
\end{align*}
\]

where \( m \in \{a, c\} \). Note that the definitions of the judgments \( \models^a \) and \( \models^c \) are mutually recursive via negation, reflecting the duality between validity and satisfiability. The strong semantics, the two judgments \( (N, i) \models^a \phi \) and \( (N, i) \models^c \phi \) are Bruns & Godefroid’s strong/thorough semantics of generalized model checking \([\text{BG00}]\) and defined as

\[
\begin{align*}
(N, i) &\models^{a+} \phi \text{ (iff for all } (I, j) \in T[N, i], (I, j) \text{ sat } \phi) \\
(N, i) &\models^{c-} \phi \text{ (iff for some } (I, j) \in T[N, i], (I, j) \text{ sat } \phi)
\end{align*}
\]

where sat is the usual satisfaction relation for Hennessy-Milner logic over labelled transition systems and \( T[N, i] \) is the class of pointed labelled transition systems that refine \( (N, i) \). Subsequently we write \( \phi \lor \psi \) for \( \neg (\neg \phi \land \neg \psi) \), \( \lor \) for its unary version, and abbreviate \( \neg \langle \alpha \rangle \neg \) by \( [\alpha] \).
Example 1.3. We verify or refute some concrete judgments $\models a$, $\models c$, $\models a^+$, and $\models c^-$, Let $N$ be the modal transition system on the left of Figure 2.

1. Since $(N, s_0) \not\models [a](\gamma) t$ and $(s_1, \alpha, s_3) \in R^c$ we have $(N, s_1) \not\models [a][\alpha](\gamma) t$.
2. By the semantics of negation and the previous item, we have $(N, s_1) \models c^- [\alpha](\gamma) t$.
3. But we also have $(N, s_1) \models c^- [\alpha](\gamma) t$ since $t_1$ in the modal transition system $M$ on the right of Figure 3 implements $s_1$ and $(M, t_1) \not\models [\alpha](\gamma) t$.
4. We have $(N, s_3) \not\models [a] (\alpha) t \lor \neg (\alpha) t$ since $(s_3, \alpha, s_3) \in R^c$ but $(s_3, \alpha, x) \in R^a$ for no $x$.
5. But we have $(N, s_3) \models a^+ (\alpha) t \lor \neg (\alpha) t$ as the latter formula is a tautology over labelled transition systems.

Modal transition systems are more expressive than labelled transition systems as they can model classes of implementations that are more general than equivalence classes of labelled transition systems with respect to bisimulation. Consequently, $\models a$ and $\models a^+$ are more powerful than $\text{sat}$ in their ability to verify goals for all implementations of a modal transition system.

The weak semantics is sound in that

$$\models a \subseteq \models a^+ \quad \text{if } \phi \text{ is asserted at } (N, i) \text{ by } \models a, \text{ then } \phi \text{ holds in all implementations of } (N, i) \quad (8)$$

$$\models c^- \subseteq \models c \quad \text{if some implementation of } (N, i) \text{ satisfies } \phi, \text{ then } (N, i) \models c^\phi$$

but loses precision. For example, for the modal transition system $N$ on the left of Figure 2 we saw that $(N, s_3) \not\models [a] (\alpha) t \lor \neg (\alpha) t$ but $(N, s_3) \not\models [a] (\alpha) t \lor \neg (\alpha) t$. This incompleteness of $\models a$ with respect to $\models a^+$ is to be expected as $(M, s) \models [a] \phi \lor \neg \psi$ attempts to reason that all implementations of $(M, s)$ satisfy $\phi \lor \psi$ by trying to reason that all implementations satisfy $\phi$ or all implementations satisfy $\psi$. On the other hand, the complexity of $\models a$ in the size of the model or formula is the one for the standard semantics $\text{sat}$ [BG00] so practitioners can get the best of both worlds, the generality of validity checking and efficiency of model checking, if they can verify $(M, s) \models a \phi$.

1.4. Outline of paper

In this first section we attempted to give an informal overview of key concepts and contributions in this paper without stressing the, admittedly, very technical nature of this work. In Section 2, we define modal transition systems and refinement formally and present a process algebra MPA whose terms have partial modal transition trees as meanings. A game semantics for modal transition systems and their refinement is presented in Section 3, characterizing refinements as winning strategies for a verifier. The universal domain $\mathcal{D}$ and its key properties are featured in Section 4 and, in Section 5, its topological structure is exploited to prove a finite-model property for abstractions in the weak and in the strong semantics. Section 6 uses the material developed in Sections 3-5 to prove the completeness of refinement. As an immediate corollary, we obtain that Hennessy-Milner logic characterizes refinement in the strong semantics as well. In Section 7 we explain that completeness of refinement means that validity checking is model checking for characteristic formulas of partial modal transition trees. Section 8 demonstrates an application of techniques developed in this paper: checking the consistency and collectively validating multiple models of the same product or design. Finally, Section 9 states related work and Section 10 concludes. In order to facilitate the progression of the narrative in this paper, we moved proofs of auxiliary or secondary results, as well as definitions and expositions of existing results from domain theory and topology, to an appendix.

2. A process algebra for partial modal transition trees

Throughout we assume a finite set of events $\text{Act}$. We define modal transition systems formally.

Definition 2.1 (Mixed [Dam96, DGG97] and modal transition systems [LT88]). A mixed transition system [Dam96, DGG97] is a triple $M = (\Sigma, R^a, R^c)$ such that, for every mode $m \in \{a, c\}$, the pair $(\Sigma, R^m)$ is a labelled transition system, i.e. $R^m \subseteq \Sigma \times \text{Act} \times \Sigma$. If $R^a \subseteq R^c$, we call $M$ a modal transition system [LT88]. A mixed transition system $(M, i)$ with a designated initial state $i$ is pointed. We call elements of $R^a$ must-transitions and elements of $R^c \setminus R^a$ may-transitions.
In this paper we assume that mixed transition systems are image-finite, unless indicated otherwise, in that for all $s \in \Sigma$, $\alpha \in Act$, and $m \in \{a, c\}$ the set $\{s' \in \Sigma \mid (s, \alpha, s') \in R^m\}$ is finite. In the definition of refinement, we work with the relational inverse of the $Q$ in [LT88, Dam96, HJS04], as done in [GJ02].

**Definition 2.2 (Refinement [LT88, Dam96] and mix condition [HJS04]).** Let $M = (\Sigma, R^a, R^c)$ be a mixed transition system.

1. A relation $Q \subseteq \Sigma \times \Sigma$ is a refinement within $M$ [LT88, Dam96] iff $\langle s, t \rangle \in Q$ implies for all $\alpha \in Act$
   (a) if $(s, \alpha, s') \in R^a$, there exists some $(t, \alpha, t') \in R^a$ such that $(s', \alpha') \in Q$;
   (b) if $(t, \alpha, t') \in R^c$, there exists some $(s, \alpha, s') \in R^c$ such that $(s', \alpha') \in Q$.

   We write $s <_M t$ or $s < t$ if there is some refinement $Q$ with $\langle s, t \rangle \in Q$. In that case $t$ refines, is abstracted by, $s$. States $s$ and $t$ are refinement-equivalent iff $s < t$ and $t < s$. We write $(M, i) < (N, j)$ if $j$ refines $i$ in the mixed transition system that is the disjoint union of $M$ and $N$. Let $\mathcal{I}[M, i]$ be the class of implementations of $(M, i)$, those mixed transition systems $(N, j)$ without any may-transitions that refine $(M, i)$.

2. We say that $M$ satisfies the mix condition (MC) iff for all $(s, \alpha, s') \in R^a$ there is some $(s, \alpha, s'') \in R^a \cap R^c$ such that $s' < s''$.

**Example 2.1.** Figure 5 reveals that mixed transition systems $(\Sigma, R^a, R^c)$ that satisfy the mix condition (MC) are refinement-equivalent to modal transition systems $(\Sigma, R^a \cap R^c, R^c)$ and so merely modal transition systems in disguise [HJS04].

Since any element of $\mathcal{I}[M, i]$ can be "colored" with any set, $\mathcal{I}[M, i]$ is not a set but a class. This has neither consequences for the results of this paper nor for practical aspects of model checking. Since the union of refinements within $M$ is a refinement within $M$, $<_M$ is the greatest refinement relation within $M$. In all respects, we identify labelled transition systems with modal transition systems that have no may-transitions, i.e. for which $R^a = R^c$. In that case, both $\models^a$ and $\models^c$ are equal to the standard satisfaction relation sat of Hennessy-Milner logic over labelled transition systems.

**Example 2.2.** The relation $Q = \{(s_1, t_1), (s_2, t_2), (s_3, t_3), (s_4, t_4), (s_5, t_5)\}$ between states of the pointed modal transition systems from Figure 2 is a refinement within their union so $s_1 < t_1$ etc.

**Definition 2.3 ([Hut04]).** The process algebra MPA is a fragment of the modal process logic in [Lar89]:

$$ p ::= 0 \mid \perp \mid \alpha_p \mid \alpha_{\perp} p \mid p + p \tag{9} $$

where $\alpha$ denotes any event in Act and all $p$ in $p + p$ are different from 0 and $\perp$. The structural operational semantics for terms of MPA is given in Figure 6. We write $\models p$ for the modal transition system derived from term $p$ in MPA and these rules.

The prefix $\alpha_{\perp}$ is the sole direct source of must-transitions whereas $\perp$ and $\alpha_p$ are the sole direct sources of may-transitions. Non-determinism $+\bot$ is the indirect source of transitions.
Fig. 6. Structural operational semantics for terms of the process algebra MPA. An expression \( p \rightarrow^\alpha p' \) denotes a may-transition from \( p \) to \( p' \) whereas \( p \rightarrow^\alpha p' \) denotes a must-transition from \( p \) to \( p' \), each labelled with some \( \alpha \in \text{Act} \). A value \( v \) stands for either \( \perp \) or \( \emptyset \). There are no transitions out of \( 0 \) and the free occurrence of \( \gamma \) ranges over all events in \( \text{Act} \).

\[
\begin{align*}
\perp & \rightarrow^\gamma \perp \\
\alpha \perp p & \rightarrow^\alpha p \\
\alpha \perp p & \rightarrow^\gamma p \\
p + q & \rightarrow^\alpha p' \\
p + q & \rightarrow^\gamma q' \\
\end{align*}
\]

Fig. 7. The operational meaning of the term \( \alpha \perp (\beta \perp + \gamma \perp) + \gamma \perp \) is a partial modal transition tree in that its leaves either deadlock \( 0 \) or model a may-stub \( \perp \) that loops for all events \( \gamma \in \text{Act} \).

Example 2.3. Figure 7 illustrates a process term \( p = \alpha \perp (\beta \perp + \gamma \perp) + \gamma \perp \in \text{MPA} \) and its operational meaning (\( \| p \|_p \)) as a partial modal transition tree.

Such meanings (\( \| p \|_p \)) are trees in that leaves either deadlock or turn into may-stubs that have may-transition loops for each \( \gamma \in \text{Act} \), so may-stubs model partiality. Partial modal transition trees are finite trees that approximate a set of infinite modal transition systems by abstracting some states and all their reachable states with \( \perp \). This idea of approximating infinite process terms or behaviors with finite ones can already be seen in the algebraic semantics à la Nivat-Courcelle-Guesarian [CN76] or à la Goguen-Thatcher-Wagner-Wright [GTWW77]. As this is elementary, we don’t show formally that every partial modal transition tree is the meaning of some term of MPA and that every such meaning is a partial modal transition tree as all summands are guarded. For example, the term \( \alpha \perp 0 + \beta \perp 0 + \perp \) is not in MPA as its rightmost summand \( \perp \) is not guarded by any prefix.

3. Refinements as winning strategies of a two-person game

Refinements of pointed modal transition systems can be characterized in terms of winning strategies of Ehrenfeucht-Fraïssé games, as worked out for labelled transition systems and bisimulation by Stirling in
Refinement is complete for implementations

[Sti96]. The idea is that checking whether $(M, i)$ is refined by $(N, j)$ can be reduced to showing that the verifier has a winning strategy in a two-person game $G[(M, i), (N, j)]$ played between a refuter, who tries to show that $(M, i)$ is not refined by $(N, j)$, and a verifier, who wants to establish that $(N, j)$ refines $(M, i)$. Since modal transition systems and refinement are generalizations of labelled transition systems and bisimulation, our adaptation of Stirling’s concepts and results to refinement of modal transition systems is straightforward.

**Definition 3.1.** Let $(M, i)$ and $(N, j)$ be two pointed modal transition systems.

1. We define a two-person game $G[(M, i), (N, j)]$ as follows.
   - **Game positions** are all pairs $(s, t)$ where $s$ and $t$ are states of $M$ and $N$, respectively;
   - there are two players, a refuter and a verifier;
   - each move consists of a question by the refuter followed, if possible, by an answer of the verifier:
     - if in position $(s, t)$ the refuter asks as question a $R^a$-transition $(s, \alpha, s')$ in $M$, the verifier has to answer with a $R^a$-transition $(t, \alpha, t')$ in $N$ resulting in the new game position $(s', t')$;
     - if in position $(s, t)$ the refuter asks as question a $R^a$-transition $(t, \alpha, t')$ in $N$, then the verifier has to answer with a $R^a$-transition $(s, \alpha, s')$ in $M$ resulting in the new game position $(s', t')$; since $R^a \subseteq R^a$, the question or answer may well be a must-transition here;
   - no other kinds of questions can be asked and only the refuter can ask questions; and
   - a run is a possibly infinite sequence of moves beginning in the initial game position $(i, j)$; the refuter wins only those runs on which the verifier eventually cannot answer; therefore, the verifier wins all infinite runs and those runs with a position in which the refuter cannot ask a question.

2. A strategy for the refuter is a partial function that maps each game position to at most one legitimate question for that position.
3. A strategy for the verifier is a partial function that maps each game position and question for that position to at most one legitimate answer to that question.
4. A strategy is winning if all runs played according to that strategy are won by the player who obeys it.

Therefore, strategies are history-free and winning strategies are total functions on positions reachable from the initial position $(i, j)$.

**Example 3.1.** 1. For the relation $Q = \{(s_1, t_1), (s_2, t_2), (s_3, t_3), (s_3, t_4), (s_3, t_5)\}$, which witnesses a refinement $s_1 \prec t_1$ of the pointed modal transition systems from Figure 2, one can easily synthesize a winning strategy for the verifier. For example, any question raised by the refuter in game position $(s_3, t_3)$ has to be a $R^a$-transition from $s_3$ labelled with $\alpha$ and the verifier responds with the $R^a$-transition from $s_3$ back to itself. The refuter cannot ask a question that involves a transition out of $s_3$ in position $(s_3, t_3)$ since there are no $R^a$-transitions out of $s_3$.

2. To see a winning strategy for the refuter, consider the game $G[(N, s_1), (M, r)]$ where $N$ is the modal transition system on the left of Figure 3 and $(M, r)$ is the pointed modal transition system from Figure 7, $r$ being the root node. The refuter begins the run by asking the $R^a$-transition $(s_1, \alpha, s_3)$ in $N$. The verifier can only reply with the $R^a$-transition labelled with $\alpha$ to the node $n$ of the right subtree since the transition to the left subtree is not in $R^a$. At game position $(s_3, n)$ the refuter can now ask the $R^a$-transition labelled with $\beta$ and source $n$, but the verifier has no matching answer in $N$ out of $s_3$.

As claimed, refinement between pointed modal transition systems is characterized by the existence of a winning strategy for the verifier in the respective refinement game.

**Theorem 3.1.** Let $(M, i)$ and $(N, j)$ be pointed modal transition systems. Then $(M, i)$ is refined by $(N, j)$ if and only if the verifier has a winning strategy in the game $G[(M, i), (N, j)]$. In that game, exactly one player has a winning strategy.

**Proof.** The proof in [Sti96] can be generalized to be aware of the presence of two kinds of transitions and to make use of the logical characterization of refinement through Hennessy-Milner logic in [Lar89, HJS04].

\[\blacksquare\]
4. A universal domain and modal transition system

One can show completeness of refinement by assuming that \((N,j)\) does not refine \((M,i)\) and constructing an implementation of \((N,j)\) that is not an implementation of \((M,i)\). Using Theorem 3.1, one could do this by synthesizing a suitable implementation from a winning strategy for the refiner in the game \(G((M,i),(N,j))\). We do not know how to do this unless \((M,i)\) and \((N,j)\) are partial modal transition trees. Therefore, we require a tool that ensures it is sufficient to consider the case of partial modal transition trees. Before we present this tool we recall concepts from domain theory [AJ94]. We refer to Appendix A for standard definitions, notation, and results from topology and domain theory used subsequently.

The mixed powerdomain \(\mathcal{M}[D]\) [Hee90, Gun92] of a SFP-domain \(D\) has as elements all pairs \((L,U)\) where \(L\) is Scott-closed and \(U\) is Scott-compact saturated such that \(L\) and \(U\) satisfy the mix condition

\[
L = \mathcal{J}(L \cap U),
\]

The order on \(\mathcal{M}[D]\) is defined by

\[
(L,U) \leq (L',U') \quad \text{iff} \quad (L \subseteq L' \& U' \subseteq U).
\]

The mix condition (10) turns out to precisely express the mix condition (MC) for mixed transition systems if, for some event \(\alpha\), we interpret elements of \(L\) as \(R^\alpha\)-successors and elements of \(U\) as \(R^c\)-successors of some state \(s\) and see the order as refinement [HJS04]: if \(s' \in L\), i.e. if \((s,\alpha,s') \in R^\alpha\), then there is some \(s'' \in L \cap U\) with \(s' \leq s''\) by (10) and so \((s,\alpha,s'') \in R^\alpha \cap R^c\). Definition 4.1 below makes all of this formal. For the sake of illustration, we state a prominent example of the mixed power domain.

**Example 4.1.** 1. [Hee90, Hee91] For \(D = \{*,\}\), \(\mathcal{L}[D] = \mathcal{U}[D] = \{\{\},\{\}\}\) as sets. The mixed powerdomain \(\mathcal{M}[D]\) contains exactly \(\{\{\},\{\}\}\), \(\{\{\},\{\}\}\), and \(\{\{\},\{\}\}\) as all three pairs satisfy the mix condition (10), whereas the pair \((\{\},\{\})\) does not. If we write \(\perp = (\{\},\{\})\), \(\ast = (\{\},\{\})\), and \(\ast' = (\{\},\{\})\), the order on \(\mathcal{M}[D]\) is the one used earlier on for refinement of models for propositional logic: \(\perp < \ast, \ast'\), the information ordering of [BG99].

2. Note that in \(\mathcal{M}([\ast])\) it is the case that \(d \leq e\) iff all maximal elements above \(e\) are also above \(d\). This is not true in every domain. For example, it is false in the domain in Figure 8 and known to be false in \(\mathcal{M}([\ast])\) [ABH97], but it is true in a domain \(D\) iff (for all \(d \in D\), the element \(d\) is the infimum in \(D\) of all maximal elements above \(d\)). We can show completeness of refinement only since our domain model is the fixed point of a domain equation, which allows us to unfold the mixed powerdomain construction in an unbound way.

**Definition 4.1 ([HJS04]).** 1. Since \(\mathcal{M}[D]\) is a SFP-domain if \(D\) is one [Hee90], we can solve the domain equation

\[
D = \prod_{\alpha \in \text{Act}} \mathcal{M}[D]
\]

where \(\prod_{\alpha \in \text{Act}}\) denotes the product over all events in \(\text{Act}\), and write \(\mathcal{D}\) for the SFP-domain and initial solution of that equation [HJS04].
2. Every element \( d \in D \) may be interpreted as a pointed mixed transition system \((D, d)\) where \( d \) is the unique initial state and the recursion \( d' = (d_\alpha', d_\gamma') \) of (12) specifies that all \( d' \) in the set \( d_\alpha' \) \( (d_\gamma') \) are exactly the \( R^a \)-successors \( (R^c \)-successors\) of \( d \) in \((D, d)\) (respectively). This makes \( D \) into a mixed transition system, which we denote subsequently by \( D = (D, R^a, R^c) \).

3. As noted in Proposition 1 in [HJS04] and as seen in Example 2.1, the mix condition (10) guarantees that the mixed transition system \( D \) is refinement-equivalent to the modal transition system \((D, R^a \cap R^c, R^c)\). Thus, all reasoning that is invariant under refinement equivalence --- as is the case in this paper --- may be done with the latter modal transition system instead of \( D \) and we abuse notation to refer to that modal transition system as \( D \) as well.

We recall some facts from [HJS04] needed as basic tools in most proofs of this paper.

**Facts 4.1.**
1. The order \( \leq \) on \( D \) is the greatest refinement in \( D \), i.e. the union of all refinements within \( D \), \( d \leq c \) in \( D \) iff \((D, d) \prec (D, c)\) by Theorem 5 in [HJS04].
2. For all pointed modal transition systems \((M, i)\) and \( \phi \) of Hennessy-Milner logic, \( (M, i) \models a \phi \) implies \( (M, i) \models a \phi \) by item 2 of Theorem 3 in [HJS04].
3. For all pointed modal transition systems without any may-transitions \( \models a \) and \( \models c \) equal the standard satisfaction relation \( \text{sat} \) over labelled transition systems by Theorem 2 in [HJS04].
4. Refinement between modal transition systems without any may-transitions coincides with bisimulation [Lar89].
5. For all pointed modal transition systems \((M, i)\) and \((N, j)\) we have \((M, i) \prec (N, j)\) iff (for all \( \phi \) of Hennessy-Milner logic, the relation \( (M, i) \models a \phi \) implies \( (N, j) \models a \phi \)) iff (for all \( \phi \) of Hennessy-Milner logic, the relation \( (N, j) \models a \phi \) implies \( (M, i) \models a \phi \)) by [Lar89] and Theorem 5 in [HJS04].
6. Every pointed modal transition system \((M, i)\) has an embedding \( \Gamma(M, i) \in D \) such that \((D, \Gamma(M, i))\) is refinement-equivalent to \((M, i)\) by item 1 of Theorem 6 in [HJS04]. In particular, the compact elements of \( D \) are precisely the embeddings of partial modal transition trees. (We discuss details of this embedding on page 13.)

**Example 4.2.**
1. Item 1 of Fact 4.1 means that we can identify the order on \( D \) with refinement on \( D \). For example, \( \bot \leq [\{0\}] \) for the denotational semantics defined below and so \((D, [\{0\}])\) refines \((D, [\bot \leq [\{0\}])\).
2. Item 2 allows us to weaken \( \models a \) judgments to \( \models c \) judgments. Let \( N \) be the model on the left of Figure 2. Then \( (N, s_2) \models a \beta \) \( \alpha \) \( \tau \) \( t \) and so \( (N, s_2) \models c \beta \) \( \alpha \) \( \tau \) \( t \) follows.
3. Item 3 can be appreciated by checking judgments \( \models m \) for the model on the right of Figure 3.
4. Item 4 follows since then \( R^c = R^a \) so the definition of refinement reads as the familiar definition of bisimulation.
5. The soundness part of item 5 is often used in this paper. Revisiting item 3 of Example 1.3 we learn that \( (M, t_1) \models \neg \gamma \alpha \) \( \tau \) \( t \) by item 3 of Fact 4.1 and so \( s_1 \prec t_1 \) implies \( (M, s_1) \models \neg \gamma \alpha \) \( \tau \) \( t \).
6. Let \( \text{Act} = (\alpha, \beta) \). If we extend MPA with recursion we can specify \( p = \alpha \) \( \beta \) \( \alpha \) \( \beta \) \( \alpha \) \( \beta \) \( \alpha \) \( \beta \) which is embedded into \( D \) as \( p_0 \) via a system of recursive equations. We order \( \alpha < \beta \) in tuples: \( 0_D = (((\{\}, \{\}), ((\{\}, \{\})), \alpha = ((\{\}, \{\}, \{\}), \{\}), \beta = ((\{0_D \}, \{0_D \}, (\{0_D \}, \{0_D \}), \{0_D \})).

In [Lar89] we find an alternative way of checking refinement by checking a system of greatest fixed-point equations with a semantics in \( \models a \). This system of equations simply expresses the refinement game as a system of formulas such that its \( \models a \) check captures the existence of a winning strategy.

**Definition 4.2.** [Lar89] Let \((M, i) = ((\Sigma, R^a, R^c, i))\) be a pointed modal transition system, not necessarily image-finite. For each \( s \in \Sigma \) we define a formula \( X_{(M, s)} \) via the greatest fixed point of the recursive equations

\[
X_{(M, s)} = (\bigwedge_{a_\alpha \in R^a} \langle a \rangle X_{(M, s')}) \land \bigwedge_{a_\alpha \in R^c} \langle a \rangle \bigvee_{a\alpha \in R^a} X_{(M, s')}
\]

for all \( s \in \Sigma \), as specified in equation (3) in [Lar89]. Intuitively, each \( X_{(M, s)} \) denotes the set of states \( t \) of \( M \) which are refinements of \( s \) within \( M \). Each equation (13) is a transducer that computes the set on the left-hand side from the sets on the right-hand side such that the logical connectives are interpreted with respect to \( \models a \) as already defined. To solve this system of equations, we initially set each \( X_{(M, s)} \) to be the
\{ 0 \} = (\{\}, \{\})_{\alpha \in \text{Act}} \quad \text{and} \quad \| \perp \| = (\{\}, \mathbb{D})_{\alpha \in \text{Act}}

(\| \alpha u. p \|_{\alpha}^{\alpha}, \| \alpha \underline{\perp}. p \|_{\alpha}^{\alpha}) = (\| \perp \|, \| p \|) \quad (\| \alpha \underline{\perp}. p \|_{\beta}^{\alpha}, \| \alpha \underline{\perp}. p \|_{\beta}^{\alpha}) = (\{\}, \{\}), \alpha \neq \beta

(\| \alpha \perp. p \|_{\alpha}^{\alpha}, \| \alpha \perp. p \|_{\alpha}^{\alpha}) = (\{\}, \| p \|) \quad (\| \alpha \perp. p \|_{\beta}^{\alpha}, \| \alpha \perp. p \|_{\beta}^{\alpha}) = (\{\}, \{\}), \alpha \neq \beta

\| p + q \|_{\gamma}^{\alpha} = \| p \|_{\gamma}^{\alpha} \cup \| q \|_{\gamma}^{\alpha} \quad \| p + q \|_{\gamma}^{\alpha} = \| p \|_{\gamma}^{\alpha} \cup \| q \|_{\gamma}^{\alpha}, \gamma \in \text{Act}.

Fig. 9. Denotational semantics \( \| p \| \) for terms \( p \) of MPA in \( \mathbb{D} \), it interprets 0 as deadlock in \( \mathbb{D} \), \( \perp \) as the least element of \( \mathbb{D} \), \( + \) as the union of \( [\text{Hoc90}] \), and prefixes as expected except for saturations with \( \perp \) and \( \underline{\perp} \) to ensure that the meaning is in \( \mathbb{D} \).

Please note that the conjunctions and disjunctions in (13) may well be infinite. Even for an image-finite pointed modal transition system, the formula \( X_{(M,i)} \) is not in general expressible in the modal mu-calculus [Koz83] via its explicit operator for greatest fixed points but it is expressible in this way if only finitely many states are \( R^* \)-reachable from \( i \) in \( M \) [Hut04].

**Example 4.3.** We write \( \nu Z. \phi \) for greatest fixed-point formulas with recursion variable \( Z \) and recursion body \( \phi \) and express \( X_{(N,s_1)} \) for the \( s_1 \) in the left of Figure 3 as a closed formula of the modal mu-calculus using greatest fixed points only. Let \( X_{(N,s_1)} \) be represented by the closed formula \( \nu Z_{s_1}[\alpha]Z_{s_1} \) of the modal mu-calculus. Let \( X_{(N,s_1)} \) be represented by the formula \( \nu Z_{s_1}[\alpha]Z_{s_1} \land [\gamma]X_{(N,s_2)} \land [\beta]Z_{s_1} \land [\gamma]X_{(N,s_3)} \) which contains \( Z_{s_1} \) as only free variable. Finally, let \( X_{(N,s_1)} \) be represented by the closed formula \( \nu Z_{s_1}[\alpha]Z_{s_1} \land [\alpha](X_{(N,s_3)} \lor X_{(N,s_2)}) \) of the modal mu-calculus. Extending the semantics of \( \models " \) to greatest fixed points via greatest semantic fixed points [HJS01] therefore captures the meaning of \( (M,i)\models " X_{(N,s_1)} \) stated in Definition 4.2.

At the end of this section, we see that \( X_{(M,i)} \) is expressible in Hennessy-Milner logic for all pointed partial modal transition trees \( (M,i) \). We now secure that \( \models " \) checks of \( X_{(M,i)} \) capture refinement checks.

**Lemma 4.1.** Let \( (M,i) \) and \( (N,j) \) be pointed modal transition systems, not necessarily image-finite. Then \( (M,i)\models (N,j)\models " X_{(M,i)} \).

**Proof.** Both statements are shown in the proof for Theorem 4.1 in [Lar89] for image-finite modal transition systems but this property is never needed in that proof, apart from the finiteness of disjunctions and conjunctions on which we do not rely here. \( \square \)

**Definition 4.3 ([Hut04]).** Figure 9 shows a denotational semantics \( \| p \| \) for terms \( p \) of MPA in \( \mathbb{D} \).

**Example 4.4.** For \( p = \alpha \perp. \perp + \beta \underline{\perp}. 0 \) and \( \text{Act} = \{ \alpha, \beta \} \), the pair \( \| p \| \) contains two pairs: \( (\{\}, \perp) \) for \( \alpha \), and \( (\| p \|, \| \perp \|) \) for \( \beta \).

We need to establish that the denotational semantics \( \| p \| \) captures the operational semantics \( \| p \| \).

**Lemma 4.2.** For all \( p \) of MPA, the pointed mixed transition systems \( \| p \|, \| p \| \) and \( \| p \| \) are refinement-equivalent.

**Proof.** We use structural induction on \( p \) and present only proofs for the two clauses for prefixes for sake of illustration. Assume that \( \| p \|, \| p \| \) and \( (\| p \|, \| p \|) \) are refinement-equivalent.

1. We show \( \| \alpha_\beta.p \|, \| \alpha_\beta.p \| \) is refinement-equivalent by induction.

- A \( R^\alpha \)-transition out of \( \| \alpha_\beta.p \|, \| \alpha_\beta.p \| \) can only arise through \( \alpha_\beta.p \rightarrow_p^\alpha p \) but \( \| \alpha_\beta.p \|, \| \alpha_\beta.p \| \) is a \( R^\alpha \)-transition in \( D \) and \( \| p \|, \| p \| \) by induction.
Refinement is complete for implementations

Fig. 10. To the left: The approximation \((N[1], t_1)\) of the pointed modal transition system \((N, t_1)\) on the right of Figure 2; for how \(t_2\) and \(t_3\) turn into m-stubs since \((t_2, t_5)\) and \((t_3, \alpha_4)\) are in \(R^*\) in \(N\). To the right: A more precise approximation \((N[2], t_1)\) of that same pointed modal transition system, which recognizes that \(t_5\) deadlocks.

- Let \((\alpha, p)[\beta, d] \in \mathbb{R}\) be a \(R^*\)-transition in \(D\). Then \(\beta = \alpha\) and \(d \in \mathbb{N}\) by induction,\((\alpha, p)[\beta, d] \in \mathbb{R}\) if \((D, p)[\beta, d] \in \mathbb{R}\) which, together with \((\alpha, p)[\beta, d] \in \mathbb{R}\), item 1 of Fact 4.1, and the transitivity of refinement implies \((\alpha, p)[\beta, d] \in \mathbb{R}\); and \((\alpha, p), \alpha, p)\) is also a \(R^*\)-transition in \(\alpha, p\).

2. Next we show \((D, \alpha, p)[\beta, d] \in \mathbb{R}\) if \((\alpha, p)[\beta, d] \in \mathbb{R}\): let \((\alpha, p)[\beta, d] \in \mathbb{R}\). Induction renders that \((D, p)[\beta, d] \in \mathbb{R}\) is refined by \((\alpha, p)[\beta, d]\) and so item 1 of Fact 4.1 and the transitivity of refinement imply \((D, p)[\beta, d] \in \mathbb{R}\); and \((\alpha, p), \alpha, p)\) is a \(R^*\)-transition.

- A \(R^*\)-transition out of \((\alpha, p), \alpha, p)\) has to arise from \((\alpha, p) \rightarrow \alpha p\) but \((\alpha, p), \alpha, p)\) is a \(R^*\)-transition and \((\alpha, p)[\beta, d] \in \mathbb{R}\) by induction.

3. We show \((\alpha, p)[\beta, d] \in \mathbb{R}\): there are no initial \(R^*\)-transitions out of \((\alpha, p)[\beta, d] \in \mathbb{R}\).

- Let \((\alpha, p)[\beta, d] \in \mathbb{R}\). Induction renders that \((D, p)[\beta, d] \in \mathbb{R}\) is refined by \((\alpha, p)[\beta, d]\) and so item 1 of Fact 4.1 and the transitivity of refinement imply \((D, p)[\beta, d] \in \mathbb{R}\); and \((\alpha, p), \alpha, p)\) is a \(R^*\)-transition.

4. Finally, we show \((D, \alpha, p)[\beta, d] \in \mathbb{R}\): there are no initial \(R^*\)-transitions out of \((\alpha, p)[\beta, d] \in \mathbb{R}\).

- A \(R^*\)-transition out of \((\alpha, p)[\beta, d] \in \mathbb{R}\) can only arise through \((\alpha, p) \rightarrow \alpha p\) but \((\alpha, p)[\beta, d] \in \mathbb{R}\) is a \(R^*\)-transition and \((\alpha, p)[\beta, d] \in \mathbb{R}\) by induction.

One can use that lemma to embed any pointed modal transition system \((N, t_1)\) faithfully into \(D\). For each \(m \geq 0\) let \((N[m], i)\) be the partial modal transition tree that unwinds the transitions beginning from \(i\) in \(N\) as a modal transition tree of depth \(m\) [HJS04]. If a leaf in that tree has a \(R^*\)-successor state in \(N\), that leaf turns into a m-stub; otherwise, said leaf deadlocks. Figure 10 depicts two of these approximations for the pointed modal transition system \((N, t_1)\) of Figure 2. For each \(m \geq 0\) there is some \(p_m\) in \(M\) such that \((N[m], i)\) is refinement-equivalent to \((\alpha, p_m)[\beta, m] \in \mathbb{R}\) as \((N[m], i)\) are pointed modal transition trees. For example, for \((N[2], t_1)\) of Figure 10 we may choose \(p_2\) in \(M\) as \((\beta, \gamma, \ldots \gamma, \ldots) + \alpha_2, \alpha_2, \ldots + \alpha_2, \alpha_2, 0\). Furthermore, the set \(\{(p_m)[m] \mid m \geq 0\}\) is directed in \(\mathbb{D}\) as \(m \leq m'\) implies \((N[m], i) \in \mathbb{R}\). For example, we have \((N[1], t_1) \in \mathbb{R}\) in Figure 10 where refinement is name identity on the common temporal layer except that may-leaves, e.g. \(t_5\) on the left, get also refined by all successor states of their name-identical version in \((N[2], t_1)\), e.g. \(t_5\) and \(t_2\) as successor states of \(t_2\) on the right. So

\[
\|N, i\| = \bigvee_{m \geq 0} p_m
\]  

(14)

exists as a directed supremum in \(\mathbb{D}\) and \((D, \|N, i\|)\) and \((N, i)\) are refinement-equivalent by item 6 of Fact 4.1.
\[
\psi_{\alpha} = \bigwedge_{\alpha \in \text{Act}} \neg \langle \alpha \rangle \tau \tau \\
\psi_{\bot} = \tau \tau \\
\psi_{\alpha \perp \rho} = \langle \alpha \rangle \psi_{\rho} \land [\alpha] \psi_{\rho} \land \bigwedge_{\beta \neq \alpha} \neg \langle \beta \rangle \tau \tau \\
\psi_{\alpha \perp \bot} = [\alpha] \psi_{\rho} \land \bigwedge_{\beta \neq \alpha} \neg \langle \beta \rangle \tau \tau \\
\psi_{p+q} = \bigwedge \{\langle \alpha \rangle \psi_{\rho} | \alpha \in \text{Act}, p + q \rightarrow \rho \} \land \bigwedge_{\alpha \in \text{Act}} \bigvee \{\psi_{\rho} | \exists \rho \in \{\bot, \tau \}: p + q \rightarrow \rho \}
\]

**Fig. 11.** The customizations of (13) for terms of MPA, where the shape of partial modal transition trees ensures that these formulas are inductively definable within Hennessy-Milner logic: For all pointed modal transition system $\langle N, i \rangle$ we have $\langle N, i \rangle \models \neg \psi_{\rho}$ iff $\langle \| p \|, p \rangle \prec (N, i)$. The expressions $\neg \langle \beta \rangle \# \tau \tau$ above result from the expression $\ldots \langle \beta \rangle \bigvee \ldots$ in (13) where the disjunction ranges over the empty set and therefore denotes $\# \tau \tau$.

From Theorem 6.4 in [Hec90] we can infer that the compact elements of $\mathcal{D}$ are all denotations of MPA:

\[\mathcal{K}(\mathcal{D}) = \{\| p \| | p \in \text{MPA}\}.\]  

(15)

In Example 4.3 we already saw that formulas $X_{\langle M, \rho \rangle}$ are expressible in the modal mu-calculus for finite-state models. For partial modal transition trees these formulas are even expressible in Hennessy-Milner logic.

**Definition 4.4.** For every $p \in \text{MPA}$ let $\psi_{p}$ of Hennessy-Milner logic be defined inductively as in Figure 11.

We ensure that $\psi_{p}$ captures $X_{\langle M, \rho \rangle}$.

**Lemma 4.3.** For all $p \in \text{MPA}$ and all pointed modal transition systems $\langle N, i \rangle$, not necessarily image-finite, we have $\langle N, i \rangle \models \# \psi_{p}$ if $\langle \| p \|, p \rangle \prec (N, i)$.

**Proof.** We prove this by structural induction on $p$.

- We have $\langle N, i \rangle \models \# \psi_{\emptyset}$ if (there are no $R^c$-transitions whatsoever out of $i$ in $N$) if $\langle \| 0 \|, 0 \rangle \prec (N, i)$.
- We have $\langle \| \bot \|, \bot \rangle \prec (N, i)$ for all pointed modal transition system $\langle N, i \rangle$.
- Using induction on $p$, we have $\langle (N, i) \rangle \models \# \psi_{\alpha}$ if (there is a $R^a$-transition $(i, \alpha, i')$ in $N$ with $\langle \| p \|, p \rangle \prec (N, i')$; all $R^c$-transitions $(i, \alpha, i'')$ in $N$ satisfy $\langle \| p \|, p \rangle \prec (N, i'')$; and there are no $R^c$-transitions out of $i$ in $N$ for other events). This exactly captures $\langle \| \alpha \# p \|, \alpha \# p \rangle \prec (N, i)$.
- By induction on $p$, we have $\langle (N, i) \rangle \models \# \psi_{\alpha \perp \rho}$ if (there are no $R^c$-transitions out of $i$ for events other than $\alpha$, and all $R^c$-transitions $(i, \alpha, i'')$ satisfy $\langle \| p \|, p \rangle \prec (N, i'')$). But this captures $\langle \| \alpha \# p \|, \alpha \# p \rangle \prec (N, i)$.
- Let $(N, i) \models \# \psi_{p+q}$. Then $(N, i) \models \# \bigwedge_{\alpha \in \text{Act}} \bigwedge_{p+q \rightarrow \rho} \langle \alpha \rangle \psi_{\rho}$ expresses that all $R^c$-transitions out of $p + q$ can be answered in the game $\mathcal{G}(\| p + q \|, p + q, (N, i))$ such that a “winning” position for the verifier is reached by induction; whereas $(N, i) \models \# \bigwedge_{\alpha \in \text{Act}} \bigvee \{\psi_{\rho} | \exists \rho \in \{\bot, \tau \}: p + q \rightarrow \rho \}$ states that all initial questions asked as $R^c$-transitions from $(N, i)$ can be answered in $\langle \| p + q \|, p + q \rangle$ to reach a “winning” position for the verifier as well, by induction. So $(N, i)$ refines $\langle \| p + q \|, p + q \rangle$.

\[\square\]

5. Abstract witnesses

Before we can prove completeness of refinement we need to understand the topology of abstraction better. In abstraction-based model checking for modal transition systems we would like a finite-model property for abstractions: if $\phi$ holds in some pointed modal transition system $\langle M, i \rangle$, there should be some finite-state
Refinement is complete for implementations

Fig. 12. A schematic description of the potential loss of precision of \( \models^a \) over \( \models^{a+} \) represented by \( V_0 \setminus \| \phi \|^a \). The properties of \( \| \phi \|^a \) and \( V_0 \) stated in this figure are shown subsequently, except for \( V_0 \cap \max(\mathbb{D}) = \| \phi \|^a \cap \max(\mathbb{D}) \) which is obvious.

\( (N,j) \) such that \( (N,j) \prec (M,i) \) and \( \phi \) holds in \( (N,j) \) as well. If this were not the case, model checking \( \phi \) for an infinite-state \( M \) through finite-state abstractions would be futile. This cannot always be secured. As Dams & Namjoshi [DN04] point out, abstraction-based model checking of the modal mu-calculus is incomplete for finite-state modal transition systems as abstractions. In this section, we prove that such a finite-model property holds for partial modal transition trees as finite abstractions of modal transition systems and Hennessy-Milner logic in the weak and in the strong semantics. For the weak semantics this is perhaps rather obvious but we strengthen it to securing that there are finitely many maximally abstract modal transition systems satisfying \( \phi \). This stronger result for the weak semantics is then used to prove the result for the stronger semantics. We define the concepts of interest via subsets of \( \mathbb{D} \).

**Definition 5.1.** Given \( \phi \) of Hennessy-Milner logic, we define

\[
\| \phi \|^a = \{ d \in \mathbb{D} \mid (D,d) \models^a \phi \}
\]

\[
\| \phi \|^c = \{ d \in \mathbb{D} \mid (D,d) \models^c \phi \}
\]

\[
V_0 = \{ d \in \mathbb{D} \mid \forall \alpha \in \max(\mathbb{D}) \subseteq \| \phi \|^a \}.
\]

The set \( V_0 \) has as elements those \( d \in \mathbb{D} \) for which all “implementations” satisfy \( \phi \) where “implementations” refers to the interpretation of that notion in the model \( \mathbb{D} \) and \( D \). Soundness of \( \models^a \) with respect to \( \models^{a+} \) therefore recognizes \( V_0 \) as a superset of \( \| \phi \|^a \). See Figure 12. For all pointed modal transition system \( (N,i) \) and \( \phi \) of Hennessy-Milner logic we have

\[ (N,i) \models^m \phi \iff \exists r \in \mathbb{C} \| \phi \|^m (m \in \{ a, c \}) \]

by item 6 of Fact 4.1. Below we show this correspondence for the strong semantics and so \( V_0 \) faithfully models implementations and the strong semantics of \( \phi \). The finite-model property for the weak semantics can surely be shown more directly than through arguments based on the model \( \mathbb{D} \). But we show a stronger property: the desired finite-state model is an abstraction; and, for each formula of Hennessy-Milner logic, there are finitely many maximally abstract models, which turn out to be partial modal transition trees. To do this, we need to recognize labelled transition systems as elements of \( \max(\mathbb{D}) \), the set of maximal elements of \( \mathbb{D} \).

**Proposition 5.1** ([Hut04]). For every labelled transition system \( (L,i) \), the element \( \| L,i \| \) is in \( \max(\mathbb{D}) \).

Now we can establish structural properties of the sets \( \| \phi \|^c \), \( \| \phi \|^a \), and \( V_0 \).

**Proposition 5.2.** Let \( \phi \) be any formula of Hennessy-Milner logic.

1. The sets \( \| \phi \|^a \) and \( \| \phi \|^c \) are Lawson-clopen in \( \mathbb{D} \).
2. There is a finite set \( P_\phi \subseteq \mathbb{D} \) such that

\[
P_\phi \subseteq \mathbb{D} \]

\[
\]
\textbullet \ \| \phi \| \equiv \mathcal{G}(\{ p \mid p \in P_{\phi} \}); \text{ and}
\textbullet \ \text{for all pointed modal transition systems } (N, i) \text{ with } (N, i) \models \phi \text{ there is some } p \in P_{\phi} \text{ with } (\| p \|, p) \prec (N, i) \text{ and } (\| p \|, p) \equiv \phi.

3. The sets \( \| \phi \| \) and \( V_{\phi} \) are Scott-open in \( \mathbb{D} \) and \( \| \phi \| \subseteq V_{\phi} \).

4. For all pointed modal transition systems \((N, i)\), we have

\[
(N, i) \models \phi \iff \| N, i \| \in V_{\phi}.
\]

The proof for item 1 above was already given in Lemma 2 of [Hut04]. We obtain a first finite-model property.

**Corollary 5.1.** For every \( \phi \) of Hennessy-Milner logic and pointed modal transition system \((N, i)\): If we have \((N, i) \models \phi\), there is some \( p \in MPA\) such that \((\| p \|, p) \prec (N, i)\) and \((\| p \|, p) \equiv \phi\).

**Proof.** If \((N, i) \models \phi\), then \(\| N, i \|\) is contained in \( V_{\phi}\) by item 4 of Proposition 5.2. The claim then follows from (15) and item 3 of Proposition 5.2 since \( \mathbb{D} \) is algebraic. \( \square \)

**Example 5.1.** 1. Unfortunately,

\[
\mathcal{J}(\mathbb{D}) \not\subseteq \mathcal{K}(\mathbb{D})
\]

so we can’t offer an easy proof that the \( p \in MPA\) in Corollary 5.1 may be chosen to be \textit{maximally} abstract: consider the pointed modal transition system \((M, i) = (\{ i \}, \{ (i, i) \})\) with \( Act = \{ \alpha, \beta \}\). In \( M \) there are no must-transitions and one may-transition \((i, \alpha, i)\) in \( R^c \) only. Then \( \| 0 \| \) is in \( \text{max}(\mathbb{D}) \cap \mathcal{K}(\mathbb{D}) \) and \( \| M, i \| \leq \| 0 \| \) but \( \| M, i \| \) is not compact as it does not equal any of its finite approximations \( \| M[m], i \| \) with \( m \geq 1 \), for may-leaves in each \( M[m] \) contain \( R^c\)-transitions for event \( \beta \).

2. There are even infinite strictly descending chains \((l_n)_{n \geq 1}\) in \( \mathcal{K}(\mathbb{D})\). For each \( n \geq 1 \) let \( p_n \in MPA\) be the term \( \alpha_{\downarrow}, \alpha_{\downarrow}, \ldots, \alpha_{\downarrow}, 0 \) which nests \( n \) may-prefixes for the same event \( \alpha \) and then stops. Each \( l_n = \| p_n \| \) is compact in \( \mathcal{K}(\mathbb{D}) \) and for all \( n < m \) we have \( l_m \leq l_n \) and \( l_n \neq l_m \).

6. **Proving completeness of refinement**

Now we can use the game-theoretic interpretation of refinement and our results from Section 5 to show that refinement is complete. As the proof of Theorem 6.1 is rather involved we first give an informal outline of strategy for this proof. In the next proposition we show that refinement is complete for partial modal transition trees as models. Then we show that completeness of refinement holds iff the domain \( \mathbb{D} \) satisfies that its order \( d \leq e \) is equivalent to \( \forall \gamma \in \text{max}(\mathbb{D}) \subseteq \mathbb{D} \cap \text{max}(\mathbb{D}) \). Next we use topology, notably the Scott-openness of \( V_{\phi} \), to argue that the equivalence of \( d \leq e \) and \( \forall \gamma \in \text{max}(\mathbb{D}) \subseteq \mathbb{D} \cap \text{max}(\mathbb{D}) \) is true for all \( d, e \in \mathbb{D} \) if it is true for \( d \) and \( e \) ranging over \( \mathcal{K}(\mathbb{D}) \) only. Finally, we note that this property restricted to elements of \( \mathcal{K}(\mathbb{D}) \) is ensured by the completeness of refinement for partial modal transition trees.

**Proposition 6.1.** For all partial modal transition trees \((M, i)\) and \((M', i')\) we have \((M, i) \prec (M', i')\) iff \( I[M', i'] \subseteq I[M, i]\).

**Proof.** Since pointed modal transition trees are denotations of terms in \( MPA \) and since \((M, i) \prec (N, j)\) implies \( I[N, j] \subseteq I[M, i]\), it suffices to show

\[
\text{"For all } p, q \in MPA \text{ the relation } I[\| p \|, q] \subseteq I[\| p \|, p] \text{ implies } (\| p \|, p) \prec (\| q \|, q)."\\(20)\]

which we do by induction on \( \text{deg}(q) \), the number of occurrences of prefixes \( \gamma_{\downarrow} \), in \( q \) summed up over all \( \gamma \in Act \). For example for the \( q \) from Figure 7 we have \( \text{deg}(q) = 3 \), stemming from the two occurrences of \( \alpha_{\downarrow} \) and one occurrence of \( \beta_{\downarrow} \).

**Base case:** Let \( \text{deg}(q) = 0 \). Then the only may-transitions, if any, in \( \| q \| \) stem from may-stubs \( \perp \). It is sufficient to show that the verifier has a winning strategy in the game \( \mathcal{G}[\| p \|, p], \| q \|, q] \). The argument is complex in that we use \( I[\| q \|, q] \subseteq I[\| p \|, p] \) to synthesize implementations on demand from which we then synthesize the promised winning strategy for the verifier. This is done by a case analysis of the players’
behavior in runs. (If there are no may-stubs $\bot$ in $q$, the arguments below reduce to stating that $(q, q, q)$ is its own implementation and therefore an implementation of $(p, p, p)$ as well.) Consider any run in the game $\Gamma([p, p], [q, q, q])$.

- Suppose that the refuter never chooses a may-transition as a question in said run. Then all her questions are must-transitions in $([p, p], [q, q])$ or $([q, q], [q, q])$. But then she cannot win this run. For let $q_1$ be obtained from $q$ by replacing all occurrences of $\bot$ in $q$ with $0$. Then $(q_1, q_1) \in \mathcal{X}([q, q], [q, q])$, since $\text{deg}(q) = 0$, and $\mathcal{X}([q, q], [q, q])$ is contained in $\mathcal{X}([p, p], [p, p])$ by assumption. Thus, $(q_1, q_1) \in \mathcal{X}([p, p], [p, p])$ and so $(p, p, p) < (q_1, q_1)$. Said run can be interpreted as a run in the game $\Gamma([p, p], [q, q, q])$ and is therefore won by the verifier.

- Suppose that the refuter does choose a may-transition as a question in said run. Let $(p', q')$ be the first game position in that run in which the refuter asks such a question, which has to be a transition in $[q, q]$ derived from $q' \rightarrow q''$ for some sub-terms $q'$ and $q''$ of $q$ and some $\alpha \in \text{Act}$ as the refuter cannot ask a may-transition in $([p, p], p)$. Since $\text{deg}(q) = 0$ we infer that $q'$ and $q''$ are the same sub-term $\bot$, where $1$ signifies this occurrence of $\bot$.

From position $(p', \bot)$ in game $\Gamma([p, p], [q, q])$ the refuter can keep asking any $R^c$-transition questions on the may-sub-$\bot$ which have to be answered by the verifier with an $R^c$-transition from $p'$ onwards in the game $\mathcal{P}$. Therefore, we can construct a winning strategy for the verifier by showing that all such answers can be given and can be chosen to reach a position $(p'', \bot)$ such that there are no $R^c$-transitions out of $p''$. This argument is inductive in the length of the path from $p'$ to $p''$, including the path of length zero, will trap the refuter in the situation of asking $R^c$-transitions as questions in $[q, q]$, and results in an infinite run won by the verifier. We write $\Sigma$ for the nary version of non-deterministic choice + in $\mathcal{P}$. 

- Let the length of the path from $p'$ to $p''$ be zero. We need to show that there are no $R^c$-transitions out of $p''$ in $[p, p]$. Let $\alpha \in \text{Act}$. Consider the implementation $([q_2, q_2])$ of $([q, q], q)$ which replaces the occurrence $\bot$ in $q$ with $(\Sigma|\not= \alpha|\beta_\alpha, 0)^3$, where $1$ still indicates the unique occurrence within the parse tree of $q_2$, and replaces all other occurrences of $\bot$ in $q$ with $0$. Note that our run in the game $\Gamma([p, p], [q_2, q_2])$ up to position $(p', \bot)$ can be interpreted as a run in the game $\Gamma([p, p], [q_2, q_2])$ as no may-transition has been asked prior to that position. But all implementations of $([q, q], q)$ are also implementations of $([p, p], p)$ by assumption. So $(p, p, p) < (q_2, q_2)$ follows and the verifier has a winning strategy in that game. This means that in position $(p', (\Sigma|\not= \alpha|\beta_\alpha, 0)^3)$ the refuter cannot ask an $R^c$-transitions labelled with $\alpha$ out of $p'$ as there are no such transitions out of $(\Sigma|\not= \alpha|\beta_\alpha, 0)^3$. Since $\alpha \in \text{Act}$ was arbitrary we conclude that there are no $R^c$-transitions out of $p''$ in $[p, p]$.

- Let $\alpha^1 \ldots \alpha^n$ be the entire sequence of events corresponding to the sequence of $R^c$-transition questions asked by the refuter from position $(p', \bot)$ towards position $(p'', \bot)$ in said run of the game $\Gamma([p, p], [q, q])$. Let $\alpha \in \text{Act}$. Consider the implementation $([q_2, q_2])$ of $([q, q], q)$ which replaces the occurrence $\bot$ in $q$ with $(\alpha^1, \alpha^2, \ldots, \alpha^n, \Sigma|\not= \alpha|\beta_\alpha, 0)^3$ and replaces all other occurrences of $\bot$ in $q$ with $0$. Our run in the game $\Gamma([p, p], [q_2, q_2])$ up to position $(p'', \bot)$ is interpretable as a run in the game $\Gamma([p, p], [q_2, q_2])$ since $R^c \subseteq R^c$. We may now reason in the same manner as done for the path of length zero.

**Inductive step**: Let the statement (20) be true for all $p, q \in \mathcal{P}^c$ with $\text{deg}(q) < n$. Now let $0 < \text{deg}(q) < n$. We denote $\text{deg}(q) = n$ with $\mathcal{I}(\mathcal{I}([q, q], q) \subseteq \mathcal{I}([p, p], p])$. Since $\text{deg}(q) > 0$ there is some $\alpha \in \text{Act}$ with some sub-term $\alpha^* \alpha^q$ in $q$. Define $q^+$ by replacing that sub-term in $q$ with $\alpha^* q^+$. Now define $q^-$ by replacing that sub-term in $q$ with $\alpha^* q^-$. If $\text{deg}(q^-) > 0$ then there is some $\alpha \in \text{Act}$ with some sub-term $\alpha^* q^-$ in $q$. We denote $\text{deg}(q^-) = m$ with $\mathcal{I}(\mathcal{I}([q, q], q^-) \subseteq \mathcal{I}([p, p], p])$. By induction on $\text{deg}(q^+)$ and $\text{deg}(q^-)$ we infer $([p, p], p) \prec (q^+, q^-)$ and $([p, p], p) \prec (q^-, q^-)$, respectively.

But then there exist some $w^+$ and $w^-$, respective winning strategies for the verifier in the two refinement games $\Gamma([p, p], [q^+, q^-])$ and $\Gamma([p, p], [q^-, q^-])$ as well. Similarly, we establish that $\mathcal{I}([q^+, q^-]) \subseteq \mathcal{I}([p, p], p)$ and $\mathcal{I}([q^-, q^-]) \subseteq \mathcal{I}([p, p], p)$ by induction on $\text{deg}(q^+)$ and $\text{deg}(q^-)$.

- As long as questions in the latter game are also questions in the game $\Gamma([p, p], [q^+, q^-])$ the verifier answers according to $w^+$ and these questions and answers will be legitimate questions and answers in the game $\Gamma([p, p], [q^-, q^-])$. We synthesize from $w^+$ and $w^-$ a winning strategy for the verifier in the game $\Gamma([p, p], [q, q])$ as follows.

- As long as questions in the latter game are also questions in the game $\Gamma([p, p], [q, q])$ the verifier answers according to $w^+$ and these questions and answers will be legitimate questions and answers in the game $\Gamma([p, p], [q, q])$ since all transitions in $\mathcal{I}([q, q])$ are also in $\mathcal{I}([q, q])$ and the same mode.
• As soon as the refuter asks a question that is not in the game $G[[p\mid p],[q^-\mid q^-]]$, that question has to be the may-transition stemming from the prefix $\alpha_\perp$, which is implemented in $q^+$ as a must-transition. In particular, the position at which this question is raised is reachable from $(p,q^+)$ in the game $G[[p\mid p],[q^+\mid q^+]]$. Hence the verifier uses $R^\perp \subseteq R^\mathsf{c}$ to interpret that question as an $R^\perp$-transition in the game $G[[p\mid p],[q^+\mid q^+]]$ and replies with the answer according to $w^+$. But this is a legitimate answer in the game $G[[p\mid p],[q\mid q]]$, resulting in a position of that game from which onwards the refuter can only reach positions and ask questions that are also positions and questions in the game $G[[p\mid p],[q^+\mid q^+]]$, respectively.

So regardless of whether the shift from $w^-$ to $w^+$ ever happens, the verifier wins every run and therefore $(|p\mid p)\prec(|q\mid q)$ follows.

As stated in the outline of our proof strategy, we need to show that $\models M', i' \cap \mathsf{max}(\mathsf{D}) \subseteq \models M, i \cap \mathsf{max}(\mathsf{D})$ captures the relation $\mathcal{I}[M', i'] \subseteq \mathcal{I}[M, i]$ for all pointed modal transition systems $(M, i)$ and $(M', i')$.

**Lemma 6.1.** For all pointed modal transition systems $(M, i)$ and $(M', i')$ we have $\mathcal{I}[M', i'] \subseteq \mathcal{I}[M, i]$ iff $\models M', i' \cap \mathsf{max}(\mathsf{D}) \subseteq \models M, i \cap \mathsf{max}(\mathsf{D})$.

In accordance with our overall strategy we now reveal that the domain is complete for refinements iff this is true for its compact elements already.

**Lemma 6.2.** Suppose that for all compact elements $k,l \in \mathsf{K}(\mathsf{D})$ we have that $\models k \cap \mathsf{max}(\mathsf{D}) \subseteq \models l \cap \mathsf{max}(\mathsf{D})$ implies $k \leq l$. Then this implication holds for all elements of $\mathsf{D}$: for all $d,e \in \mathsf{D}$ the relation $\models d \cap \mathsf{max}(\mathsf{D}) \subseteq \models e \cap \mathsf{max}(\mathsf{D})$ implies $d \leq e$.

**Proof.** Let $d,e \in \mathsf{D}$ with $\models d \cap \mathsf{max}(\mathsf{D}) \subseteq \models e \cap \mathsf{max}(\mathsf{D})$. Choose any $k \in \mathsf{K}(\mathsf{D})$ below $d$. As $\mathsf{D}$ is algebraic, it suffices to show $k \leq e$. By (15) there is $p \in \mathsf{MPA}$ with $k = \models p \models$ and so $e \in V_{\psi_p}$ by Lemma 4.3 since $\models k \cap \mathsf{max}(\mathsf{D}) \subseteq \models e \cap \mathsf{max}(\mathsf{D})$ and $\models k \subseteq \models e$ since $k \leq d$. But $\mathsf{D}$ is algebraic and the set $V_{\psi_p}$ is Scott-open by item 3 of Proposition 5.2. So there is some $l \in \mathsf{K}(\mathsf{D})$ below $e$ with $l \in V_{\psi_p}$. Thus, $\models k \cap \mathsf{max}(\mathsf{D}) \subseteq \models l \cap \mathsf{max}(\mathsf{D})$ which implies $k \leq l$ by the assumption of the lemma. So $k \leq l \leq e$ renders $k \leq e$.  

Finally, we can tie together all arguments to secure the completeness of refinement.

**Theorem 6.1.** Refinement for pointed modal transition systems is complete with respect to the interpretation of pointed modal transition systems as their respective classes of implementations: for all pointed modal transition systems $(M, i)$ and $(N, j)$ we have $(M, i) \prec (N, j)$ iff $\mathcal{I}[N, j] \subseteq \mathcal{I}[M, i]$.

**Proof.** From Lemma 6.1 we know that the completeness of refinement holds iff for all $d,e \in \mathsf{D}$ the relation $\models d \cap \mathsf{max}(\mathsf{D}) \subseteq \models e \cap \mathsf{max}(\mathsf{D})$ implies $d \leq e$. By Proposition 6.1, Lemma 6.1 applied to partial modal transition trees, and (15) this holds for all compact elements $d,e \in \mathsf{K}(\mathsf{D})$. By Lemma 6.2, this is sufficient.

Using the completeness of refinement, we show that refinement is characterized by Hennessy-Milner logic not only under the weak, but also under the strong semantics.

**Corollary 6.1.** For all pointed modal transition systems $(M, i)$ and $(N, j)$ we have $(M, i) \prec (N, j)$ iff (for all $\phi$ of Hennessy-Milner logic, $(M, i) \models^{\mathsf{c}} \phi$ implies $(N, j) \models^{\mathsf{c}} \phi$) iff (for all $\phi$ of Hennessy-Milner logic, $(N, j) \models^{\mathsf{c}} \phi$ implies $(M, i) \models^{\mathsf{c}} \phi$).

Using the results of Jagadeesan & Schmidt in [GJ03], we infer that refinement is complete for a host of 3-valued models and their notion of refinement, including those that combine event and state information.

**Corollary 6.2.** Refinement is complete for implementations for the model-checking frameworks of Kripke modal transition systems [HJS01] and partial Kripke structures [BG99].

**Proof.** Jagadeesan & Godefroid show [GJ03] that there are LOGSPACE and PTIME translations between any of these models and modal transition systems, and between Hennessy-Milner logic and the corresponding temporal logics for the other models such that refinement and the meaning of model checks is preserved and reflected. Thus our arguments are invariant under such a change of representation. 

7. Completeness of refinement as semantic minimization

This paper asks as second question whether a semantic minimization $\phi \mapsto \phi^+$ exists for modal transition systems and Hennessy-Milner logic:

“For all $\phi$ of Hennessy-Milner logic, is there some $\phi^+$ of Hennessy-Milner logic such that for all pointed modal transition systems $(M, i)$: $(M, i) \models^{=\phi} \phi$ iff $(M, i) \models^{=\phi^+}$?” (21)

As Blamey has shown [Bla80], the answer is affirmative in the setting of 3-valued propositional logic where $\models$ is Kleene’s strong 3-valued interpretation of propositional logic, our weak semantics, and $\models^{=\phi}$ the supervaluational meaning [VF66], our strong semantics. As one would expect for propositional logic, the length of $\phi^+$ is exponential in the length of $\phi$ in the worst case.

Example 7.1. Let the formula $\phi$ be $p \leftrightarrow q$, which states that $p$ and $q$ have the same truth value. Then $\phi$ is semantically self-minimizing in that we may choose $\phi^+$ to be $\phi$. For if $M$ is any 3-valued model for which $\phi$ evaluates to $\bot$ in the weak semantics, there will always be two refining 2-valued models for which $\phi$ evaluates to different truth values.

The connection between semantic minimization and completeness of refinement can now be explained. In Lemma 4.3 we showed that for all formulas $\psi_p$ of Figure 11 and for the operational meaning $(\llbracket p \rrbracket, p)$ of the process term $\phi$ defined in Figure 6 we have

$$(\llbracket p \rrbracket, p) \prec (M, i) \iff (M, i) \models^{=\phi_p} ((M, i) \text{ pointed modal transition system}).$$

Moreover, the set of all such $\psi_p$ logically characterizes refinement by (15) and items 1 and 5 of Fact 4.1 in the weak semantics. We introduce terminology for the case in which $\models^{=\phi}$ in Figure 12 is a proper subset of $V_\phi$.

Definition 7.1. A formula $\phi$ of Hennessy-Milner logic loses precision iff for some pointed modal transition system $(M, i)$, $(M, i) \models^{=\phi} \phi$ but $(M, i) \not\models^{=\phi}$.

Remark 7.1. The completeness of refinement is equivalent to the fact that no formula $\psi_p$ loses precision. To see this we repeatedly used (22), Lemma 4.3, and the fact that $\models^{=\phi}$ equals sat on implementations:

- Assume that some $\psi_p$ loses precision. Then there is some $(M, i)$ with $(M, i) \models^{=\phi^+} \psi_p$ and $(M, i) \not\models^{=\phi^+} \psi_p$. The former guarantees $\mathcal{I}[M, i] \subseteq \mathcal{I}[p \llbracket, p]$, the latter means $\mathcal{I}[p \llbracket, p] \not\prec (M, i)$, and so refinement is incomplete.

- Conversely, assume that no $\psi_p$ loses precision. Proof by contradiction: Assume that $\mathcal{I}[M, i] \subseteq \mathcal{I}[N, j]$ and $(N, j) \not\prec (M, i)$. By items 1 and 6 of Fact 4.1 and (15) there is some $p$ with $(\llbracket p \rrbracket, p) \prec (M, i)$ such that $\mathcal{I}[M, i] \subseteq \mathcal{I}[N, j]$ and $(N, j) \not\prec (M, i)$. From $\mathcal{I}[M, i] \subseteq \mathcal{I}[N, j]$ and $\not\prec (M, i)$ we get $(M, i) \models^{=\phi^+} \psi_p$ as valid $\models^{=\phi^+}$ checks are preserved under refinement by item 5 of Fact 4.1. Thus, $(M, i) \models^{=\phi^+} \psi_p$, as well as $\psi_p$ does not lose precision. But $(M, i) \models^{=\phi^+} \psi_p$ means $(\llbracket p \rrbracket, p) \prec (M, i)$, a contradiction.

That is, completeness of refinement is equivalent to the fact that $\psi_p^+$ of (21) exists for all formulas $\psi_p$ of Figure 11 and that $\psi_p^+$ may be chosen to be $\psi_p$; these formulas are semantically self-minimizing and so validity checks $(M, i) \models^{=\phi^+} \psi_p$ are reducible to model checks $(M, i) \models^{=\phi^+} \psi_p$. This insight corroborates that answering question (4) is non-trivial since the semantics of $\psi_{p+q}$ under $\models^{=\phi}$ depend heavily on the evaluation of disjunctions, potentially causing a loss of precision.

8. Checking consistency and verifying goals of multiple models

Before we conclude, we point out a genuine application of our results in the realm of requirements engineering. Basic functional requirements may be captured as a term $p$ of a process algebra, say, MPA for sake of simplicity. In fact the process algebra MPA, enriched with parallel composition, is sufficient for activities such as state-space exploration, simulation, and bounded model checking [BCCZ99] where one puts bounds on the depth of computation paths.
As requirements are likely to be under-specified, prefixes come in two modes, $\alpha, \beta$, and $\alpha \perp$, as in the process algebra $MPA$. A considerable problem in practice is that there are often finitely many such terms $p_1, p_2, \ldots, p_n$ each describing requirements on, or aspects of, the same design or product. This raises fundamental problems:

1. Events known in one term may be foreign in another.
2. Does the finite set of terms $\{p_k \mid 1 \leq k \leq n]\}$ have a common implementation?
3. If so, do all (respectively some) common implementations of that set satisfy a goal?

Example 8.1. Figure 13 depicts two modal transition systems that have a common refinement. The reader is invited to determine whether all such common refinements satisfy that there is an event path on which $\beta$ events happen until one gets to a state at which not all $\alpha$ events lead to states at which $\beta$ can happen.

The first problem could be solved by identifying and encoding all foreign events as may-transitions that can lead from states to subsets of states with that imported foreign label, creating a common set of events for all terms $p_k$. Two obvious choices of such subsets are the entire state space, modelling state-wide divergence, and singleton self-loops, modelling that local state is unaffected by foreign events.

The second problem may now be solved as follows. For each $1 \leq k \leq n$ let $(\| p_k \|, p_k)$ be the pointed modal transition system that arises as the operational meaning of the term $p_k$. We assume that each term $p$ has a formula $\psi_p$ of some logic, not necessarily Hennessy-Milner logic or the modal mu-calculus, satisfying (22) for a judgment $\models^\omega$ that interprets conjunction in a compositional manner. A common implementation is a pointed labelled transition system that refines all of these pointed modal transition systems, therefore being a witness to the satisfiability of

\[
\bigwedge_{k=1}^{n} \psi_{p_k}. \tag{23}
\]

Conversely, any labelled transition system that is a satisfiability witness of that formula has to be a common implementation of all of the $(\| p_k \|, p_k)$ by (22). For $MPA$, the formulas in (23) can be computed inductively as in Figure 11 and satisfiability for Hennessy-Milner logic and labelled transition systems is PSPACE-complete, for this is the case for the basic modal logic $K$ and Kripke structures and there are linear-time and log-space translations between these frameworks [GJ03]. These results are modest since practical specifications often require recursion and so the $\psi_{p_k}$ in (23–27) need to be replaced with the $X_{(\| p_k \|, p_k)}$ in (13) which are then expressible in the modal mu-calculus if $\| p_k \|$ is finite-state.

For $MPA$, the third problem above can be specified in terms of $\models^\omega$ and $\models^{\omega+}$ by exploiting the topological structure of the $SFP$-domain $\mathbb{D}$. Each $\psi_{p_k}$ is of the form $\mathcal{C}_k$ for a finite set $C_k \subseteq \mathcal{K}(\mathbb{D})$ by Proposition 3.2. Since the finite intersection of Scott-open, Scott-compact sets is again Scott-compact and Scott-open in a $SFP$-domain [AJ94] we conclude that

\[
\bigcap_{k=1}^{n} \psi_{p_k} \models^\omega = \mathcal{C} (\{ q \}) \mid q \in Q \}
\]

for some finite set $Q \subseteq MPA$. The set $Q$ is non-empty iff the first problem has a positive solution. In that case, each $(\| q \|, q)$ with $q \in Q$ is a maximally abstract common refinement of all $(\| p_k \|, p_k)$, up to refinement.
equivalence, and every common refinement of all \( (\emptyparallel p_k \mid \emptyparallel q \mid) \) is a refinement of some \( (\emptyparallel q \mid) \) with \( q \in Q \). Therefore, we can answer whether all common refinements satisfy \( \phi \) by answering

\[
\text{"Does} (\emptyparallel q \mid) \models \phi \text{ hold for all} q \in Q?\" \tag{25}
\]

A computationally less expensive approximation uses \( \models \) in (25) instead of \( \models \).

No doubt has the reader realized that this constitutes a solution only in as much as one is able to compute all \( q \in Q \) from the set of all \( (\emptyparallel p_k \mid \emptyparallel q \mid) \). In the general case, we may have to rely on the validity of (22) to decide whether all common implementations of all \( p_k \) satisfy \( \phi \) by checking the satisfiability of

\[
\neg \phi \land \bigwedge_{k=1}^{n} \psi_{p_k} \tag{26}
\]

over labelled transition systems and negating the answer of that check. Dually, in asking whether some common implementation of all \( p_k \) satisfies \( \phi \), we check whether

\[
\phi \land \bigwedge_{k=1}^{n} \psi_{p_k} \tag{27}
\]

is satisfiable by some labelled transition system. The check for consistency in (23) is a special case of (27) where \( \phi \) equals \( \top \) and consistency checks should, intuitively, be in PTIME and not PSPACE-complete or EXPTIME-complete as is the case for checks of general Hennessy-Milner logic or modal mu-calculus formulas, respectively.

All this is modest but promising progress in a longstanding open problem in formal software engineering and needs to be strengthened with algorithmic support in future work.

9. Related work

Dams & Namjoshi [DN04] show that finite-state modal transition systems are incomplete as abstractions of infinite-state modal transition systems for modal mu-calculus checking. They propose focused transition systems as a generalization of modal transition systems, show completeness for this class of models, and define game semantics for refinement of such systems and model checks of alternating tree automata on such systems.

In [Hut04] further structural properties of refinement are shown, notably, that \( \text{max}(D) \) is Lawson-closed in \( D \), that \( X = \text{max}(D) \) is a Stone space where the topology has basis \( \{ k \cap \text{max}(D) \mid k \in K(D) \} \), and that the space \( X \) is a topological model of all labelled transition systems over a finite set of events Act up to bisimulation such that the embeddings of image-finite labelled transition systems are dense in \( X \). Consistency measures for modal transition systems are introduced and discussed.

The paper [HJS04] presents the SFP-domain \( D \) and its underlying modal transition system and provides most of the basic facts on which the work in this paper relies. Although we have strived to make this paper self-contained, we recommend reading [HJS04].

Uchitel & Chechik [UC04] merge modal transition systems with overlapping but different sets of events to obtain a minimal common refinement and suggest user participation to explore common behavior if no minimal common refinement exists.

Bruns & Godefroid develop 3-valued model checking in [BG99]; their generalized model checking in [BG00] eradicates any loss of precision through automata-theoretic means that blow up the model to be checked. They offer model-checking algorithms and complexity bounds for most practically relevant temporal logics. The abstraction-based approach to 3-valued model checking is described in [GHJ01] and, by Godefroid & Jagadeesan, in [GJ02].

Dams [Dam96] develops an abstract-interpretation and partition-refinement framework for model checking based on mixed transition systems which are often not modal transition systems since optimality considerations suggest to construct as few may-transitions and as many must-transitions as possible. It is unknown whether the models in [Dam96] always satisfy the mix condition (MC).
The proof that semantic minimization, in the sense of (21), is possible for all formulas of propositional logic is implicitly contained in Blaney's thesis [Bla80]. An implementation of such minimizations, using prime implicants and binary decision diagrams, was given by Reps et al. in [RLS02].

The definition of modal transition systems and their refinement is given by Larsen & Thomsen in [LT88]. Larsen defines a semantics for Hennessy-Milner logic over modal transition systems and shows that it logically characterizes refinement in [Lar89]; in loc. cit. he also proposes modal specifications as a way of combining modal transition systems declaratively.

Cousot & Cousot invent abstract interpretation [CC77] as a formal framework in which one can express abstractions of concrete data and transformations, and formulate soundness and optimality principles for abstract interpretations of concrete transformations. In that context, non-distributive flow analyzes [NNH99] lose precision in a way similar to the loss encountered by the weak semantics of Hennessy-Milner logic.

Van Fraassen defines the super-valuational meaning of propositional logic formulas in [vF66].

10. Conclusions

We presented the 3-valued model-checking framework for modal transition systems, their refinement, and a weak and a strong semantics for Hennessy-Milner logic. The weak semantics, a bottom-up labelling algorithm for model checks, is well known to lose precision. The strong semantics does not lose precision but known algorithms require the transformation of the model under check. We asked whether one model refines another if and only if its implementations are also implementations of the system it is refining. We answered this affirmatively and thus showed that Hennessy-Milner logic characterizes refinement under the strong semantics. This also means that the characteristic formulas of partial modal transition trees do not lose precision when checked under the weak semantics.

The proofs of these results relied in part on a topological model developed with Jagadeesan & Schmidt in [HJS04], a SFP-domain which is also a universal modal transition system. Using this model, we furthermore secured that all model checks can be decided by model checks on partial modal transition trees that abstract the model under check; and that, in the weak semantics, there are finitely many models — which happen to be partial modal transition trees — that are maximally abstract with respect to satisfying a given formula of Hennessy-Milner logic. For the strong semantics we proved that each model that satisfies a formula of Hennessy-Milner logic is abstracted by a partial modal transition tree that satisfies that formula. These results constitute an abstraction-based finite-model property for the weak and the strong semantics.

We presented some preliminary results on applications of this work in the context of determining whether finitely many pointed modal transition systems have a common refinement and, if so, whether all common refinements satisfy a goal.

Finally, we remarked that the results of this paper are stable under a change of representation as they apply to 3-valued models that are state-based or combine state and event information.

Acknowledgments

We wish to thank Samson Abramsky, Michael Goldsmith, Chris Hankin, Achim Jung, Bill Roscoe, and especially Glenn Bruns, Patrice Godefroid, Radha Jagadeesan, and David Schmidt for their insight and feedback. Discussions with Sebastian Uchitel triggered the development in Section 8. We also thank the anonymous referees for making many suggestions on improving the organization of this paper and the clarity of its presentation.

References


Refinement is complete for implementations


A. Definitions and established results from topology and domain theory

(We recommend [AJ94] for a thorough reference on these issues.) A partial order \((D, \leq)\) is a set \(D\) with a binary relation \(\leq\) on \(D\) that is reflexive, transitive, and antisymmetric. An upper bound for a subset \(A\) of a partial order \((D, \leq)\) is an element \(u \in D\) such that \(a \leq u\) for all \(a \in A\). We write \(ub(A)\) for the set of upper bounds of \(A\). The set \(mub(A) = \{u \in ub(A) \mid \forall d \in D\; d \leq u \& d \in ub(A) \Rightarrow d = a\}\) consists of all minimal upper bounds of \(A\) in \(D\).

A subset \(A\) of \(D\) is directed iff all finite subsets of \(A\) have an upper bound in \(A\). Given \(X \subseteq D\) we write \(\downarrow_{X} D\) for \(\{d \in D \mid \exists x \in X\; d \leq x\}\), \(\uparrow_{X} D\) for \(\{d \in D \mid \exists x \in X\; x \leq d\}\), and elide the subscript \(D\) if it is determined by context. We use \(\downarrow x\) and \(\uparrow x\) if \(X = \{x\}\). Subsets \(U\) of \(D\) with \(U = \uparrow U\) are upper sets, and subsets \(L\) of \(D\) with \(L = \downarrow L\) are lower sets.

A partial order \((D, \leq)\) is a dcpo iff all its directed subsets have a least upper bound \(\bigvee A\), i.e. iff there is some \(\bigvee A \in D\) with \(ub(A) = \bigvee A\). An element \(k \in D\) is compact in a dcpo \(D\) iff for all directed sets \(A\) of \(D\) with \(k \leq \bigvee A\) there is some \(a \in A\) with \(k \leq a\); we write \(K(D)\) for the set of compact elements. A dcpo \(D\) is algebraic iff for all \(d \in D\) the set \(\{k \in K(D) \mid k \leq d\}\) is directed with least upper bound \(d\). For a finite subset \(F\) of \(D\) define \(mub^{1}(F) = mub(F)\), \(mub^{n+1}(F) = mub(mub^{n}(F))\) for all \(n \geq 1\), and \(mub^{\omega}(F) = \bigcup_{n \geq 1} mub^{n}(F)\). A SFP-domain is an algebraic dcpo \(D\) such that for every finite subset \(F \subseteq K(D)\) the set \(\text{mub}^{\omega}(F)\) is finite and contained in \(K(D)\) with \(ub(F) = \uparrow \text{mub}(F)\).

A topological space \((X, \tau)\) consists of a set \(X\) and a family \(\tau\) of subsets of \(X\) such that \(\{\}\) and \(X\) are in \(\tau\), and \(\tau\) is closed under finite intersections and arbitrary unions. Elements \(O \in \tau\) are \(\tau\)-open, complements \(X \setminus O\) with \(O \in \tau\) are \(\tau\)-closed, and sets that are \(\tau\)-open and \(\tau\)-closed are \(\tau\)-clopen. A topological space \((X, \tau)\) is \(\tau\)-compact iff for all \(U \subseteq \tau\) with \(X \subseteq \bigcup U\) there is a finite subset \(F \subseteq U\) with \(X \subseteq \bigcup F\). A subset \(A\) of \(X\) is dense in \((X, \tau)\) iff \(A \cap O\) is non-empty for all non-empty \(O \in \tau\).

Given a topological space \((X, \tau)\) and a subset \(Y \subseteq X\), the subspace topology on \(Y\) consists of the set \(\{O \cap Y \mid O \in \tau\}\). A subset \(Y\) of \(X\) is \(\tau\)-compact iff \(Y\) is compact in its subspace topology. A subset \(Y\) of \(X\) is \(\tau\)-saturated in \(X\) iff \(Y\) is the intersection of \(\tau\)-open sets. The upper powerdomain \(U[X]\) is defined as the set of all \(\tau\)-compact \(\tau\)-saturated subsets of \(X\), ordered by reverse inclusion.

The definitions and characterizations below assume that \(D\) is a SFP-domain. The Scott-topology on \(D\) consists of all subsets \(U\) of \(D\) satisfying \(U = \uparrow (U \cap K(D))\); such elements are Scott-open. The Lawson-topology on \(D\) consists of all subsets \(V\) of \(D\) such that \(x \in V\) implies the existence of some \(k, l \in K(D)\) with \(x \in K(D)\). Note that every Scott-open is Lawson-open and every Scott-closed is therefore Lawson-closed. For all \(d \in D\), the set \(\downarrow d\) is Lawson-closed upper. A subset \(C\) of \(D\) is Scott-compact (Scott-)saturated in \(D\) iff \(C\) is Lawson-closed upper in \(D\). A subset \(U\) of \(D\) is Scott-open and Scott-compact iff \(U\) is of the form \(\downarrow F\) for a finite set \(F \subseteq K(D)\).

A collection \((F_{i})_{i \in I}\) of subsets of \(D\), indexed by a directed set \((I, \leq)\), is filtered iff (for all \(i, j \in I\) there is some \(k \in I\) with \(k \in ub(\{i, j\})\)) such that \(F_{k} \subseteq F_{i} \cap F_{j}\). The Hofmann-Mislove Theorem [HM81] states that if the intersection \(\bigcap_{i \in I} C_{i}\) of a filtered collection of Scott-compact saturated sets \((C_{i})_{i \in I}\) in \(D\) is contained in a Scott-open set \(U \subseteq D\), then there is some \(i_{0} \in I\) with \(C_{i_{0}} \subseteq U\) already.
B. Proofs of auxiliary or secondary results

Proof of Proposition 5.1 Let $(L, i)$ be a labelled transition system, a modal transition system with
must-transitions only, and set $d = \{L, i\}$. Since $(L, i)$ and $(D, d)$ are refinement-equivalent by item 6 of Fact 4.1
and since $[\phi]^a = [\phi]^c$ on modal transition systems with must-transitions only by item 3 of Fact 4.1, we infer
that, for all $\phi$ of Hennessy-Milner logic, $(D, d)[\phi] = \lceil \phi \rceil$ if $(D, d)[\phi] = \lceil \phi \rceil$. Proof by contradiction: If $d \not\leq \text{max}(D)$,
there is some $e \in D$ with $d \leq e$ and $e \not\leq d$. Since $D$ is algebraic, the latter implies that there is some
$k \in K(D)$ with $k \leq e$ and $k \not\leq d$. By (15) there is some $p \in MPA$ with $k = \{ p \}$. But $k \leq e$ means
$(D, e)[\phi] = \lceil \phi \rceil$ and $k \not\leq d$ means $(D, d)[\phi] = \lceil \phi \rceil$. By Lemma 4.3 which, as inferred above, implies $(D, d)[\phi] = \lceil \phi \rceil$. But
$(D, e)[\phi] = \lceil \phi \rceil$ implies $(D, e)[\phi] = \lceil \phi \rceil$ by item 2 of Fact 4.1 and so $d \leq e$ implies $(D, d)[\phi] = \lceil \phi \rceil$ by item 5 of Fact 4.1,
a contradiction. 

Proof of Proposition 5.2 1. We proceed by structural induction on $\phi$ of Hennessy-Milner logic. This is evident
for the clauses $t$, negation, and conjunction since $D$ is Lawson-clopen and duplicens are closed under set
composition $[A \phi] = [B \phi] \cap [A \phi]$ and $[A \phi] = [B \phi] \cup [A \phi]$ and finite intersections. We require
mode-dependent proofs for $(\alpha, \phi)$, where $[\langle \alpha \rangle \phi] = \{ d \in D \mid d \phi \phi \cap K = \phi \neq \{ \} \}$. For $m = \{ a, c \}$.

1. Let $m = a$. By Theorem 4.2 in [HJS04] all $[\phi] = [\phi] = [\phi]$ for a finite subset $F_0 \subseteq K(D)$ as $D$ is algebraic. By the definition of
$[\langle \alpha \rangle \phi]$ we have $e \in [\langle \alpha \rangle \phi]$ if $e_0 \cap F_0 \neq \emptyset$ and $e_0 \cap F_0 \neq \emptyset$ (as $e_0$ is a lower set). For each
$y \in F_0$ define $c(y) = (c(y), \cdots, \alpha \in A)$ by $c(y, \cdots, \alpha \in A)$ for all $\beta \neq \alpha$, and $c(y, \cdots, \alpha \in A) = (y, \cdots, \alpha \in A, \cdots)$. Then
$C = \{ c(y) \mid y \in F_0 \}$ is finite and $C \subseteq K(D)$. Since $y \in F_0$, we get $\forall \subseteq [\langle \alpha \rangle \phi]^a$ as the latter set is upper. Note that for each $y \in F_0$ we have $c(y) \leq e$ in $D$ iff $y \in e_0$. Therefore,
$[\langle \alpha \rangle \phi] = [\phi] = [\phi]$ implies $[\forall \subseteq [\langle \alpha \rangle \phi]$. Thus, $[\langle \alpha \rangle \phi] = [\phi] = [\phi]$.

2. The first claim follows from item 1 and the fact that all $[\phi] = [\phi] = [\phi]$ as shown in [HJS04]. The second claim follows from this, Lemma 4.2, and item 6 of Fact 4.1.

3. By item 2 of Theorem 4 in [HJS04] each $[\phi] = [\phi]$ is Scott-open. The inclusion $[\phi] = [\phi]$ follows from item 5 of Fact 4.1. The set $V_0$ is an upper set in $D$ since $d \in V_0$ and $d \leq e$ imply $\exists \subseteq \max(D) \subseteq [\phi] = [\phi]$ and so $e \in V_0$. Let $d \in V_0$. Since $D$ is algebraic, it suffices to show that $l \in V_0$ for some $l \in [\phi] = [\phi]$ as then $V_0$ is Scott-closed. Let $M$ be the intersection of all Lawson-closed upper subsets of $D$ that contain $\max(D)$. Then for every $x \in D$ the set $\exists \subseteq \max(D) = [\phi]$ is Scott-closed and saturated in $D$ as the intersection of two Lawson-closed upper sets. Since $D$ is algebraic, we infer that $\exists \subseteq \max(D) = [\phi]$ contains $\max(D)$ and is a filtered collection of Scott-closed saturated subsets of $D$ with intersection equal to $\exists \subseteq \max(D)$. But the latter set is contained in $[\phi] = [\phi]$. This is so since $[\phi]$ is Lawson-closed upper and $B \cap \exists \subseteq \max(D) = [\phi]$ that contains $\max(D)$. Then for every $x \in D$ the set $\exists \subseteq \max(D) = [\phi]$ contains $\max(D)$ and is a filtered collection of Scott-closed saturated subsets of $D$ with intersection equal to $\exists \subseteq \max(D)$. But the latter set is contained in $[\phi] = [\phi]$ that contains $\max(D)$. This is so since $[\phi]$ is Lawson-closed upper and $B \cap \exists \subseteq \max(D) = [\phi]$ that contains $\max(D)$.

4. First let $\{ N, i \}$ be in $V_0$ and consider any $(M, j)$ in $\forall \subseteq \max(D)$, which has an element
$\exists \subseteq \max(D)$ and so $\exists \subseteq \max(D)$ is in $\forall \subseteq \max(D)$ implies $(D, [\phi]) = [\phi]$ whence $(M, j) = [\phi]$ by item 6 of Fact 4.1. Thus,
$(N, i) = [\phi]$ follows.

Second, assume that $(N, i) = [\phi]$. Proof by contradiction: If $\{ N, i \} \not\subseteq V_0$, then there has to be some $m \in \exists \subseteq \max(D)$ satisfying $\exists \subseteq \max(D)$. Let $X(N, i)$ be the formula defined in (13) which satisfies, for all pointed
modal transition systems $(M,j)$, that $(M,j)\models^a X_{(N,i)}$ iff $(N,i)\prec (M,j)$. In particular, $(D,m)\models^a X_{(N,i)}$. Therefore, $(D,m)$ is a labelled transition system, not necessarily image-finite, that satisfies $\neg \phi \land X_{(N,i)}$. Since $\neg \phi$ is in Hennessy-Milner logic and since all bodies in any $X_{(N,i)}$ with $i'$ being $R^2$-reachable from $i$ in $N$ involve finite disjunctions or conjunctions only — as $(N,i)$ is image-finite — there is some image-finite labelled transition system $(L,l)$ that satisfies $\neg \phi \land X_{(N,i)}$ as well. But then $(L,l)\models^a X_{(N,i)}$ implies $(N,i)\prec (L,l)$ and so $(L,l)$ is an implementation of $(N,i)$ which does not satisfy $\phi$, a contradiction.

**Proof of Lemma 6.1** Given $\forall M',i' \in \max(D) \subseteq \forall M,i \in \max(D)$, item 6 of Fact 4.1 and Proposition 5.1 imply $I[M',i'] \subseteq I[M,i]$. Conversely, we use proof by contradiction. Suppose that $\forall M',i' \in \max(D) \not\subseteq \forall M,i \in \max(D)$ and $I[M',i'] \not\subseteq I[M,i]$. The former means that there is some $m \in \forall M',i' \in \max(D)$ that is not in $\forall M,i \in \max(D)$, i.e. $\forall M,i \not\subseteq m$. Since $D$ is algebraic, there is some $p \in MPA$ with $\| p \| \not\subseteq \| M,i \|$ and $\| \{ p \} \not\subseteq m$ by (15).

Since $m \in \forall M',i'$, we obtain $(D,m)\models^a X_{(M',i')}$ by Lemma 4.1. But $\| p \| \not\subseteq m$ implies $(D,m)\not\models^c \psi_p$ by Lemma 4.3 and so $(D,m)\models^a \neg \psi_p$ as $m \in \max(D)$ so $(D,m)$ is refinement-equivalent to a pointed labelled transition system by Theorem 2.2 of [Hut04] and $\models^a$ equals $\models^c$ for $(D,m)$. This implies $(D,m)\models^a \neg \psi_p \land X_{(M',i')}$ as $\models^a$ is sound and so there is some labelled transition system that satisfies $\neg \psi_p \land X_{(M',i')}$ but for all states $s$ of $(M',i')$ the bodies in $X_{(M',i')}$ are finite conjunctions and disjunctions — as $(M',i')$ is image-finite — and $\neg \psi_p$ is a formula of Hennessy-Milner logic, so there has to be some image-finite labelled transition system $(L,l)$ satisfying $\neg \psi_p \land X_{(M',i')}$ as well. In particular, $(L,l)$ satisfies $X_{(M',i')}$ and so $(L,l) \in I[M',i']$ which is contained in $I[M,i]$ by assumption. Therefore, $\{ p \} \not\subseteq \{ M,i \} \subseteq \{ L,l \}$ implies that $(L,l)$ satisfies $\psi_p$, a contradiction.

**Proof of Corollary 6.1** We write (1), (2), and (3) for the first, second, and third “iff” statement (respectively). Then (1) $\Rightarrow$ (2) holds as $(M,i)\prec(N,j)$ implies $I[N,j] \not\subseteq I[M,i]$. The implication (2) $\Rightarrow$ (3) follows from the duality of $\models^a$ and $\models^c$ as the statements quantify over all $\phi$ of Hennessy-Milner logic. To see (3) $\Rightarrow$ (1), we use proof by contradiction. Assume (3) and let $(M,i) \not\prec (N,j)$. By Theorem 6.1, there is some $(L,l) \in I[N,j] \setminus I[M,i]$. Then $(M,j) \not\prec (L,l)$ implies that there is some $p \in MPA$ with $\{ p \} \not\subseteq \{ M,i \}$ and $\{ p \} \not\subseteq \{ L,l \}$ by (15). But then $\{ p \} \not\subseteq \{ M,i \}$ implies $(D,\{ M,i \})\models^a \neg \psi_p$, since $\forall M,i \in \max(D) \subseteq \forall \{ p \} = \neg \psi_p$, whereas $\{ p \} \not\subseteq \{ L,l \}$ implies $(D,\{ L,l \})\models^c \neg \psi_p$ and so $(D,\{ N,j \})\models^c \neg \psi_p$, as $(L,l) \in I[N,j]$. But then $(D,\{ N,j \})\models^c \neg \psi_p$ follows by duality and so (3) implies $(D,\{ M,i \})\models^c \neg \psi_p$, a contradiction to $(D,\{ M,i \})\models^a \psi_p$.

\[ \square \]}