A Minimal Hybrid Logic for Intervals

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Abstract

Taking our inspiration from van Benthem's treatment of temporal interval structures, and Halpern and Shoham's work on intervals, we introduce an interval hybrid temporal logic with two binary relations, precedence and inclusion, for talking about interval temporal structures. This paper can be seen as an continuation of the work began in an earlier paper, in which we undertook a purely modal treatment of interval temporal structures. By introducing an interval hybrid temporal logic, we enrich the logic with *nominals*, and thereby increase the expressivity of the logic. We study the interval hybrid temporal logic in its full generality and identify two important classes of interval temporal structures: the class of minimal interval structures, and the class of van Benthem minimal interval structures. We present sound and complete tableau calculi for both classes of structures. We prove that the logic of minimal interval structures is decidable, by developing a novel bulldozing technique that handles both the presence of nominals and the interaction between the two relations. We go on to show that the satisfiability problem is EXPTIME-complete. We conclude the paper with the remark that the decidability (or otherwise) and complexity of the logic of van Benthem minimal interval structures remains an interesting open problem.

Keywords: tableau method, interval logics, hybrid logic, bulldozing

1 Background and Motivation

An interval-based approach to temporal reasoning has always had its adherents. And whilst, no doubt, the philosophical problems inherent in a conception of time as consisting of durationless moments, have weighed heavily on the minds of the early pioneers, much of the impetus for subsequent research has come from computer science, and in particular artificial intelligence. Here it has been advocated that interval-based representations of time are simpler and more natural in formalizing common sense reasoning than the standard scientific models [11].

The formal study of temporal logic was initiated by the philosopher Arthur Prior [23], it was first applied in theoretical computer science to reason about programs [20]. The initial perspective was point-based: formulas were interpreted over time points, and the temporal structures were typically assumed to be discrete. Subsequent research in temporal logic began to concentrate on intervals rather than points. Again, the initial impetus to deal with intervals rather than points came from the philosophers [10, 14, 16, 25]. In computer science, work began on *process logic* [15, 19, 22], where intervals (or "paths") represent pieces of computation, and *interval temporal logic* [12]. A concise review of these earlier interval logics is given in [13].

However, when we adopt an interval-based perspective, we have some very basic decisions to make about what are intervals, about what are the natural relations between intervals, and about any restrictions there should be on the valuation of atoms. In this paper the choices we will make will be motivated by previous work on intervals; in particular it builds

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on both the philosophical and computer science perspective gained from the work of van Benthem [25] and Halpern and Shoham [13].

1.1 The logic of Halpern and Shoham

In [13], Halpern and Shoham present an interval logic HS which can be viewed as a generalisation of point-based modal temporal logic. HS is a temporal logic with the following modalities: $\langle B \rangle$, $\langle E \rangle$, $\langle A \rangle$, $\langle \overline{B} \rangle$, $\langle \overline{E} \rangle$, $\langle \overline{A} \rangle$, which have the following intended readings:

$\langle B \rangle \phi$ ϕ holds at a strict beginning interval of the current one
$\langle E \rangle \phi$ ϕ holds at a strict end interval of the current one
$\langle A \rangle \phi$ ϕ holds at an interval met by the current one, i.e., it begins where the current one ends
$\langle \overline{B} \rangle \phi$ ϕ holds at an interval which has the current one as a beginning interval
$\langle \overline{E} \rangle \phi$ ϕ holds at an interval which has the current one as a ending interval
$\langle \overline{A} \rangle \phi$ ϕ holds at an interval meeting the current one

For the semantics, they opt for temporal structures $(T, \leq)$ where T is a set of points and $\leq$ is a partial order on T. Intervals are then defined as (convex) sets of points. Their choice of modalities suffice to capture the 13 possible relations between distinct intervals in a linear temporal structure, as is illustrated in Figure 1. They go on to show that for most interesting classes of temporal structures, validity and satisfiability is undecidable. One of the results they establish states the following:

FACT 1.1
The validity problem for each of the following classes of temporal structures is r.e.-complete:

1. the class of all temporal structures.
2. the class of all linear temporal structures.
3. the class of all discrete temporal structures.
4. the class of all dense temporal structures.
5. the class of all dense, linear, unbounded temporal structures.

A *complete* temporal structure is one in which any sequence with an upper bound has a least upper bound; a class of temporal structures is said to be complete if *all* structures in the class are complete [13]. For classes that are complete as well as containing an infinitely ascending sequence, they show that the validity problem is even harder. One of the interesting open problems they raise at the end of the paper asks: *what happens to the complexity of the validity problem if we modify the logic, in particular what happens for weaker or incomparable combinations of modal operators*?

1.2 The logic of van Benthem

In [25], van Benthem gives a more abstract treatment of intervals. He does this by (1) combining several Allen relations into just two relations; and (2) by passing from sets of points to abstract objects. He introduces the following structure for talking about intervals:

DEFINITION 1.2
A period structure $\mathcal{F}$ is an ordered triple $\langle I, \sqsubseteq, < \rangle$ of a non-empty set I carrying two binary relations $\sqsubseteq$ ('inclusion') and $<$ ('precedence').

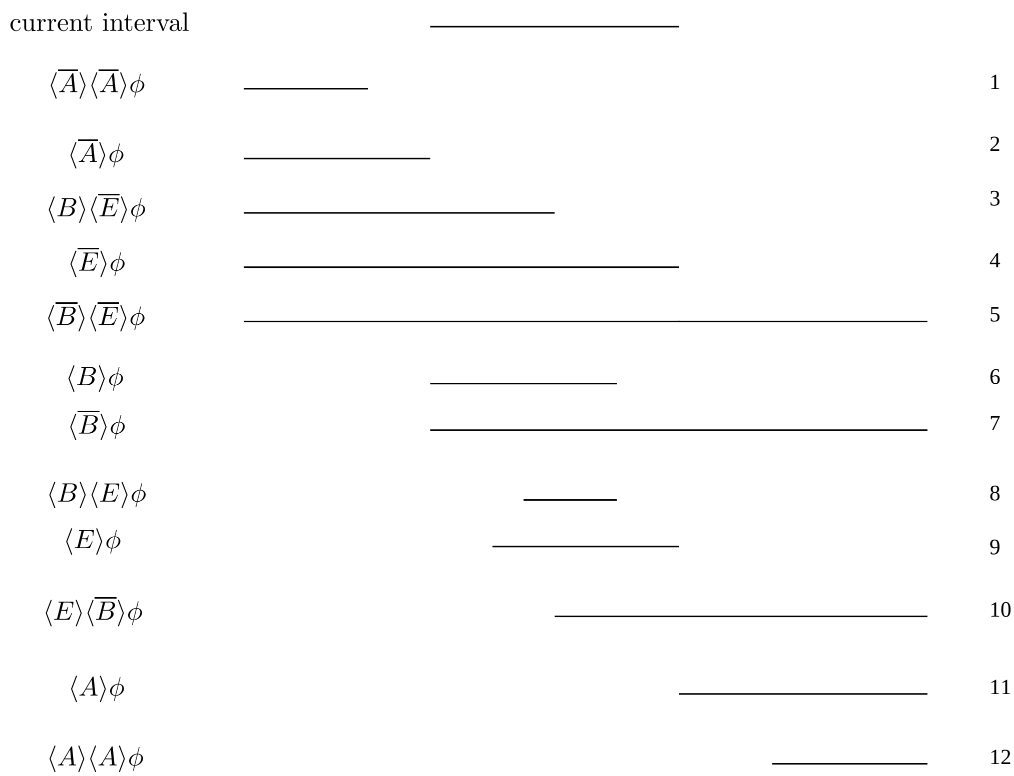

Fig. 1.

As we have seen in the case of Halpern and Shoham, intervals are built over temporal structures $(T, \leq)$, where T is a set of points and $\leq$ is a partial ordering. As a consequence of intervals being a set of points of a partial order, the precedence relation between intervals will be transitive; similarly, inclusion between intervals will also be transitive. Furthermore, the following conditions will hold between intervals x, y, z:

$\forall xy(x < y \rightarrow \forall u(u \sqsubseteq x \rightarrow u < y))$ (Right Monotonicity)
$\forall xy(x < y \rightarrow \forall u(u \sqsubseteq y \rightarrow x < u))$ (Left Monotonicity)

These properties are basic in the sense that they arise from sets of points of a partial order. It is therefore natural to adopt them as axioms in the abstract setting. What additional properties we should adopt is motivated by the choices van Benthem makes.

Whilst van Benthem's treatment of intervals is more abstract, his choices, like Halpern and Shoham's, are informed by concrete examples; in his case the choice of the minimal basis for period structures is informed by his investigation of intervals over $\mathbb{Z}$ and $\mathbb{Q}$.

He proposes that $<$ be a strict ordering, and $\sqsubseteq$ be a partial ordering. However, because his intention is to axiomatise intervals over $\mathbb{Z}$ and $\mathbb{Q}$, he also insists on the following condition:

$\forall xy(xOy \rightarrow \exists z(z \sqsubseteq x \wedge z \sqsubseteq y \wedge \forall u((u \sqsubseteq x \wedge u \sqsubseteq y) \rightarrow u \sqsubseteq z))$ (Conjectivity),
where $xOy =_{df} \exists z(z \sqsubseteq x \wedge z \sqsubseteq y)$.

This condition states that any two overlapping intervals have a greatest common subinterval. It is questionable whether this property should be adopted as a basic property of interval structures, since it is not always valid if intervals are sets of points of a partial order.

A period structure in which $<$ is a strict ordering, $\sqsubseteq$ is a partial ordering, and monotonicity (left and right) and conjectivity are satisfied will be called a *van Benthem minimal interval structure*. We also identify a simpler class, which we call *minimal interval structures*, in which the conjectivity condition is dropped. While van Benthem studied some instances of these two classes, the Halpern and Shoham question of whether complexity of the logic drops if we pass to a more general setting remained open.

1.3 Initial choices for our logic

In attempting to give a positive answer to the Halpern and Shoham question, a good strategy for obtaining a simpler logic, in complexity terms, is to abstract and have less properties. As we have seen, the work of van Benthem provides clear guidelines on how to go about achieving this.

In this paper, it is our intention to study the logic of intervals in its full generality, and we hope to show that the choices we make give rise to an interval logic with a simple syntax and semantics and which has a good computational complexity.

Ontology. Are intervals primitive objects in the logic, or are they defined in terms of points? In philosophy, you find logics of both kind. In computer science, almost all interval-based logics construct intervals from points (with Allen's logic [2] and the Event Calculus [18] being the only exceptions we are aware of). Because intervals as derived objects have been extensively studied in computer science, and because we believe treating intervals as first class citizens is worthy of consideration, we will join the minority by taking intervals as primitive objects.

Commitment to an underlying temporal structure. Most interval-based temporal logics in computer science have been committed to the discrete and linear view of time. Our logic (like the HS logic) will be quite general in this respect: we study two general classes of interval temporal structures, *minimal interval structures* and *van Benthem minimal interval structures*. By imposing only the most elementary constraints on our logic, we will not exclude branching and linear time, dense and discrete time, bounded and unbounded time, and so on.

Choice of operators. The strong commitment to a discrete and linear order, in computer science, dictated fairly standard modal operators. In philosophy, there has been less uniformity. Following van Benthem [25], we employ two very natural pairs of modal operators, precedence and inclusion for talking about interval temporal structures.

Evaluating formulas. An issue that arises when evaluating propositions in computer science is whether or not *locality* is assumed; a logic is local if a propositional atom is true over an interval iff it is true over its starting point. In philosophy, an assumption sometimes made is that of *homogeneity.* A logic is homogeneous when, roughly speaking, a proposition is true over an interval iff it is true over all its subintervals. By taking intervals as primitives, we do not assume either homogeneity or locality.

The main results we establish concerning our two classes of interval temporal structures are the following:

- Soundness and Completeness of a tableau system for the class of minimal interval structures,
- Soundness and Completeness of a tableau system for the class of van Benthem minimal interval structures,

- Decidability of the logic of minimal interval structures,
- EXPTIME-completeness of the satisfiability problem for the logic of minimal interval structures.

The decidability and relatively good computational complexity of our logic can be attributed to two main differences between our approach and that of Halpern and Shoham:

1. Our notion of what is an interval is weaker than theirs.
2. Our choice of modalities is expressively weaker than theirs.

Their notion of an interval is very different from ours. They build intervals over a (convex) set of points; they further restrict themselves to considering only *linear intervals*, which means that for any two points t_1 and t_2 such that $t_1 \leq t_2$, the set of points $\{t : t_1 \leq t \leq t_2\}$ is totally ordered; and they insist that the set of intervals is closed under certain operations, e.g., if $\langle x, y \rangle$ is an interval, then $\langle x, x \rangle$ is also an interval. By taking intervals as primitives, we are not required to make any of the above suppositions. Thus, the classes of temporal structures that they investigate are less general than the ones we consider. And because our notion of an interval is weaker than theirs, our modalities are also weaker in their expressivity. For example, it is not clear how we should express the $\langle B \rangle$ modality in our logic, since this operator uses the notion of a *starting point*.

While it could be argued that minimal interval structures are too weak to be considered as 'real' interval temporal structures, nonetheless, any 'real' interval temporal structure would satisfy the minimal constraints that we impose. As such we believe that it is worthwhile and instructive to begin our investigation of interval temporal structures by considering this general class. The question of what, if any, further assumptions are needed to obtain 'real' interval temporal structures we leave for future work.

1.4 Hybrid vs. Modal

What benefits do we gain from using hybrid logic, as opposed to modal logic, to study interval temporal structures? In [17], we introduced an interval temporal logic and gave a purely modal treatment of interval temporal structures. We briefly summarise the main results of that paper: (1) we proved that the satisfiability problem for the class of minimal interval structures was PSPACE-complete; (2) we showed that the logic of minimal interval structures was the *same* as the logic of van Benthem minimal interval structures, and (3) we highlighted some important limitations in the expressivity of the logic; in particular, we showed that the *Difference* operator was not definable. In this paper, we enrich the logic with nominals, which can be simulated by the difference operator. The use of hybrid logic with its ability to name states, and 'jump' to states named by nominals (via the @-operator), brings many advantages to applications for which intervals are a natural formalism. For example, in the planning domain, the use of nominals allows us to locate each task individually. The extra logical apparatus that hybrid logics affords us allows us to formulate properties not expressible in standard modal logic, e.g., irreflexivity and antisymmetry; and, unlike in the modal case, it allows us to distinguish the logic of minimal interval structures from the logic of van Benthem minimal interval structures (cf. Example 5.11). However, the advantages gained by using hybrid logic are mitigated by the increase in the computational complexity of the satisfiability problem for the logic of minimal interval structures. In [17], the satisfiability problem was shown to be PSPACE-complete for the interval temporal language; in this paper, we show that the satisfiability problem for the logic of minimal interval structures is EXPTIME-complete for the interval hybrid temporal language.

1.5 Structure of the paper

In sections 2 - 5, we define an interval hybrid temporal logic and give a (sound and complete) tableau calculus for both the class of minimal interval structures, and the class of van Benthem minimal interval structures. We also demonstrate that the two logics are distinct. Sections 6-8 contain the body of technical results concerning the logic of minimal interval structures. In section 6, we prove a general truth lemma that establishes, for any formula ϕ, if ϕ is satisfiable in a model based on a minimal interval structure, then ϕ is satisfiable in a finite model based on a pre-interval structure. In section 7, we obtain the necessary decidability result by showing how a minimal interval structure can be obtained from a pre-interval structure. In the process, we develop a robust bulldozing technique that handles the presence of nominals. In section 8, we show that the satisfiability problem for the class of minimal interval structures in the interval hybrid language is EXPTIME-complete. We conclude the paper in section 9.

2 Syntax and Semantics of an interval hybrid temporal logic ($\mathcal{IHL}$)

In their simplest form, hybrid languages are modal languages which use formulas to refer to specific points in a model. Hybridisation is about handling different types of information in a uniform way. Given a basic modal language built over propositional variables $\Phi = \{p, q, r, \ldots\}$, let $\Omega = \{i, j, k, \ldots\}$ be a nonempty set disjoint from Φ. The elements of Ω are called *nominals*; they are a second sort of atomic formula which will be used to name states. We call $\Phi \cup \Omega$ the set of *atoms* and define the *interval hybrid logic* $\mathcal{IHL}$ (over $\Phi \cup \Omega$) as follows:

$$\phi ::= i \mid p \mid \neg\phi \mid \phi \wedge \phi \mid \Diamond\phi \mid @_i\phi$$

where $\Diamond \in \{\langle D\rangle, \langle U\rangle, \langle F\rangle, \langle P\rangle\}$, $i \in \Omega$ and $p \in \Phi$. We define $\Box\phi = \neg\Diamond\neg\phi$. Also the boolean connectives $\top$, $\bot$, $\rightarrow$ and $\vee$ are defined in the standard way.

Interpretation is carried out using the *Kripke satisfaction definition*. This is defined as follows. Let $\mathcal{M} = (W, R_>, R_<, R_\sqsupseteq, R_\sqsubseteq, V)$ where $w \in W$, $R_>$, $R_<$, $R_\sqsubseteq$, $R_\sqsupseteq$ are binary relations on W such that $R_<$ is the converse of $R_>$, and $R_\sqsubseteq$ is the converse of $R_\sqsupseteq$, and $V : \Phi \cup \Omega \rightarrow \mathbb{P}(W)$. We insist that for all $i \in \Omega$, $V(i)$ is a *singleton* subset of W. Then:

$\mathcal{M}, w \models p$ iff $w \in V(p)$
$\mathcal{M}, w \models \neg\phi$ iff $\mathcal{M}, w \not\models \phi$
$\mathcal{M}, w \models \phi \wedge \psi$ iff $\mathcal{M}, w \models \phi$ and $\mathcal{M}, w \models \psi$
$\mathcal{M}, w \models \langle D\rangle\phi$ iff $\exists w'(wR_\sqsupseteq w'$ and $\mathcal{M}, w' \models \phi)$
$\mathcal{M}, w \models \langle U\rangle\phi$ iff $\exists w'(wR_\sqsubseteq w'$ and $\mathcal{M}, w' \models \phi)$
$\mathcal{M}, w \models \langle F\rangle\phi$ iff $\exists w'(wR_< w'$ and $\mathcal{M}, w' \models \phi)$
$\mathcal{M}, w \models \langle P\rangle\phi$ iff $\exists w'(wR_> w'$ and $\mathcal{M}, w' \models \phi)$
$\mathcal{M}, w \models i$ iff $w \in V(i)$
$\mathcal{M}, w \models @_i\phi$ iff $\mathcal{M}, w' \models \phi$, where w' is the denotation of i under V.

We will sometimes use (mostly in our proofs) $\sqsupseteq$, $\sqsubseteq$, $<$, $>$ as abbreviations for $R_\sqsupseteq$, $R_\sqsubseteq$, $R_<$, $R_>$ where there is no likelihood of confusion.

Definition 2.1
We say that $\mathcal{F} = \langle W, R_<, R_>, R_\sqsubseteq, R_\sqsupseteq \rangle$ is a Minimal Interval Structure if it satisfies the following conditions: for $<$ (and its converse, $>$) *Irreflexivity* and *Transitivity*; for $\sqsubseteq$ (and its converse, $\sqsupseteq$) *Reflexivity*, *Transitivity* and *Antisymmetry*, plus the following two interaction axioms: *Right Monotonicity* and *Left Monotonicity* (cf. Definition 1.2).

Van Benthem's conjectivity condition is dropped from our definition of a minimal interval structure for two reasons. First, unlike in the modal case [17], the logic of van Benthem's minimal interval structures *is* different from the logic of minimal interval structures, as we shall later demonstrate; it is therefore worthwhile considering the more general class in its own right. Secondly, while we give a completeness result for the class of van Benthem minimal interval structure, we have to admit that it remains an open question whether the logic is decidable.

As *concrete* examples of minimal interval structures, we give the following examples (the second example will be used for the bulldozing in section 7):

1. Let $W = \{[n, m] \mid n \leq m \text{ and } n, m \in \mathbb{Z}\}$,
   $R_\sqsubseteq = \{([n, m], [k, l]) \mid k \leq n \leq m \leq l\}$,
   $R_\sqsupseteq = \{([n, m], [k, l]) \mid n \leq k \leq l \leq m\}$,
   $R_< = \{([n, m], [k, l]) \mid n \leq m < k \leq l\}$,
   $R_> = \{([n, m], [k, l]) \mid k \leq l < n \leq m\}$.
2. Let $U = \{(n, m) : n < m \text{ and } n, m \in \mathbb{Q}\}$,
   $R'_\sqsubseteq = \{((n, m), (k, l)) : k \leq n < m \leq l\}$,
   $R'_\sqsupseteq = \{((n, m), (k, l)) : n \leq k < l \leq m\}$.
   $R'_< = \{((n, m), (k, l)) : n < m \leq k < l\}$,
   $R'_> = \{((n, m), (k, l)) : k < l \leq n < m\}$.

It is straightforward to check that the above structures are minimal interval structures. We will show that right monotonicity holds for example 1. So, suppose that $[a, b]R_\sqsubseteq[c, d]$ and $[c, d]R_<[n, m]$. We want to show that $[a, b]R_<[n, m]$. By definition we have that $c \leq a \leq b \leq d$ and $c \leq d < n \leq m$, and therefore $a \leq b < n \leq m$. This implies that $[a, b]R_<[n, m]$, as we had to show. Indeed, it can be easily checked that in fact the conjectivity condition is also satisfied by the above two examples.

We will also have recourse to the following class of structures, which will play a very important technical role throughout the course of this paper, particularly in relation to the decidability and complexity results.

Definition 2.2
We say $\mathcal{F} = \langle W, <, >, \sqsubseteq, \sqsupseteq \rangle$ is a **pre-interval structure** if it satisfies the following conditions: for $<$ (and its converse, $>$) *Transitivity*; for $\sqsubseteq$ (and its converse, $\sqsupseteq$) *Reflexivity*, *Transitivity*, plus the two interaction axioms: *Right Monotonicity* and *Left Monotonicity* (cf. Definition 1.2).

For pre-interval structures we no longer insist that the precedence relation is *irreflexive*, nor that the inclusion is *antisymmetric*.

3 Tableau System (TS) for the interval hybrid temporal logic

In this section we present a set of tableau rules for the *interval hybrid temporal logic*. In the subsequent two sections we prove soundness and completeness for the tableau system with respect to the class of minimal interval structures. Every formula in the tableau is of the form $@_s\phi$ or $\neg @_s\phi$, and such statements are called *satisfaction statements*.

DEFINITION 3.1
A Tree is a structure $(T, Succ)$, where T is a non-empty set and $Succ$ is a binary relation on the elements in T such that, for $x, y \in T$:

1. $\forall x \exists_{\leq 1} y$ *Succ(y,x)*.
2. There is a unique x such that $\forall y \neg$*Succ(y,x)*.
3. $Succ^*$ (the transitive closure of *Succ*) is well-founded.

A Tableau is $(T, Succ, \lambda)$, where $(T, Succ)$ is a tree and λ assigns a non-empty set of satisfaction statements $\lambda(t)$ to each $t \in T$. A branch of a tableau is a maximal subset of T that is linearly-ordered by $Succ^*$. We say a branch ρ 'contains' a formula ψ if $\exists t \in \rho(\psi \in \lambda(t))$.

Roughly speaking, tableau methods are search procedures that work by systematically exploring all possible consequences of an assumption in the search for a counter-example. A tableau is a well-founded tree whose nodes are labelled by formulas and which is built via certain tableau rules (for breaking logical formulas down to simpler formulas). In our presentation of the tableau rules, the formula above the horizontal line is the input to the rule. For example, in the $[\neg\wedge]$-rule $@_s(\phi \wedge \psi)$ is assumed to be false, so one of ϕ or ψ must be false at s; the step of the tableau development for this formula will split into two branches, one with the added assumption that ϕ is false at s, and the other with the added assumption that ψ is false at s. A rule such as $\neg\wedge$ is called a *branching rule* because it yields two alternative outputs.

We will now give the set of tableau rules for the *interval hybrid temporal logic*. In what follows ϕ and ψ denote formulas, and s, t, u and a denote nominals. First, the rules for the booleans:

$$\frac{@_s\neg\phi}{\neg @_s\phi}[\neg] \qquad \frac{\neg @_s\neg\phi}{@_s\phi}[\neg\neg]$$

$$\frac{@_s(\phi \wedge \psi)}{@_s\phi, @_s\psi}[\wedge] \qquad \frac{\neg @_s(\phi \wedge \psi)}{\neg @_s\phi \mid \neg @_s\psi}[\neg\wedge]$$

Now we incorporate rules for the satisfaction operators and the mechanism needed to formulate modal *theories* of state equality and state succession:

$$\frac{@_s @_t\phi}{@_t\phi}[@] \qquad \frac{\neg @_s @_t\phi}{\neg @_t\phi}[\neg @]$$

$$\frac{\text{[s on branch]}}{@_s s}[Ref] \qquad \frac{@_s t}{@_t s}[Sym] \qquad \frac{@_s t, @_t\phi}{@_s\phi}[Nom] \qquad \frac{@_s\Diamond t, @_t u}{@_s\Diamond u}[Bridge]$$

(by '*s on branch*' in the statement of *Ref* we simply mean that some formula on the branch in question contains an occurrence of s).

Now for some interesting rules, those dealing with the modalities (where $\Diamond = \{\langle D\rangle, \langle U\rangle, \langle P\rangle, \langle F\rangle\}$):

$$\frac{@_s\Diamond\phi}{@_s\Diamond a, @_a\phi}[\Diamond] \qquad \frac{\neg @_s\Diamond\phi, @_s\Diamond t}{\neg @_t\phi}[\neg\Diamond]$$

(Where a is a new nominal). The $\Diamond$-rule is called an *existential* rule. It is the only rule that introduces new nominals into a tableau.

We also need the following two rules to establish that $\langle D\rangle$ and $\langle U\rangle$ are each others converse, and likewise for $\langle P\rangle$ and $\langle F\rangle$:

$$\frac{@_s\langle U\rangle t}{@_t\langle D\rangle s}[C(\sqsupseteq)] \quad \frac{@_s\langle D\rangle t}{@_t\langle U\rangle s}[C(\sqsubseteq)] \quad \frac{@_s\langle P\rangle t}{@_t\langle F\rangle s}[C(<)] \quad \frac{@_s\langle P\rangle t}{@_t\langle F\rangle s}[C(<)]$$

In order to capture the minimal interval structure we need the following rules for *inclusion*:

$$\frac{@_s s}{@_s\langle D\rangle s}[Ref(\sqsubseteq)] \quad \frac{@_s\langle D\rangle t, @_t\langle D\rangle u}{@_s\langle D\rangle u}[Trans(\sqsubseteq)] \quad \frac{@_s\langle D\rangle t, @_t\langle D\rangle s}{@_s t}[Antisy(\sqsubseteq)]$$

With respect to the *precedence* relation we will need the following rules:

$$\frac{@_s s}{\neg @_s\langle F\rangle s}[Irref(<)] \qquad \frac{@_s\langle F\rangle t, @_t\langle F\rangle u}{@_s\langle F\rangle u}[Trans(<)]$$

Note that while the input to the [*Ref*]- and the [*Irref*]-rules may seem superfluous, we insist on having the inputs in order to ensure termination for the tableau.

Finally, for the interaction between the modalities we have:

$$\frac{@_s\langle U\rangle t, @_t\langle F\rangle u}{@_s\langle F\rangle u}[Mon(\sqsubseteq, <)] \qquad \frac{@_s\langle U\rangle t, @_t\langle P\rangle u}{@_s\langle P\rangle u}[Mon(\sqsubseteq, >)]$$

4 Soundness and systematic construction

We turn now to soundness and completeness results for the class of minimal interval structures. This brief section lays the groundwork for the completeness results that follow; its main purpose is to define a notion of systematic tableau construction general enough to establish completeness for countable languages.

Definition 4.1 (Consistency and provability)
A branch of a tableau is closed iff it contains some satisfaction statement and its negation, and a branch which is not closed is open. A tableau is closed iff all its branches are closed. A formula ϕ is provable iff there is a closed tableau whose root node is $\neg @_i\phi$ (here i can be

any nominal not occurring in ϕ) and ϕ is consistent iff $\neg\phi$ is not provable. A set of formulas Σ is consistent iff for any finite subset Σ^f of Σ, the conjunction of all the formulas in Σ^f is consistent.

Recall that every formula in a tableau is of the form $@_s\phi$ or $\neg @_s\phi$ and that such formulas are called satisfaction statements. Suppose that Σ is a set of satisfaction statements, and that R is one of our tableau rules. Then:

1. If R is *not* a branching rule, and R takes a single formula as input, and Σ^+ is the set obtained by adding to Σ all the formulas (there are at most two) yielded by applying R to $\sigma_1 \in \Sigma$, then we say that Σ^+ is a result of expanding Σ by R .
2. If R is a binary rule (for instance, the $\neg\Diamond$-rule), then Σ^+ is the set obtained by adding to Σ the formula yielded by applying R.
3. If R is the $\neg\wedge$-rule and Σ^+ is a set obtained by adding to Σ the formula yielded by one of the two possible outcomes of applying R to $\sigma_1 \in \Sigma$, then we say that Σ^+ is a result of expanding Σ by R.
4. If a nominal s belongs to some formula in Σ, then $\Sigma \cup \{@_s s\}$ is a result of expanding Σ by Ref.

Definition 4.2 (Satisfiable by label)
Suppose that Σ is a set of satisfaction statements and that $\mathcal{M} = (W, R_<, R_>, R_\sqsubseteq, R_\sqsupseteq, V)$ is a standard interval model. We say that Σ is *satisfied by label* in $\mathcal{M}$ if and only if for all formulas in Σ:

1. If $@_s\phi \in \Sigma$, then $\mathcal{M}, w \models \phi$; and
2. If $\neg @_s\phi \in \Sigma$ then $\mathcal{M}, w \not\models \phi$.

(Here w is the denotation of s under V). We say that Σ is *satisfiable by label* if and only if there is a standard model in which it is satisfied by label.

Lemma 4.3
Suppose Σ is a set of satisfaction statements that is satisfiable by label. Then, for any R, at least one of the sets obtainable by expanding Σ by R is satisfiable by label.

Proof. The only non-trivial cases are when R is an existential rule. But even these are straightforward, as we now show for the case of $\langle D\rangle$ modality. Suppose Σ is satisfiable by label, this means there is a standard model in which it is satisfied by label. Let $@_s\langle D\rangle\phi \in \Sigma$. So we have some standard model $\mathcal{M}$ such that $\mathcal{M}, w \models \langle D\rangle\phi$ (where w is the denotation of s under V). This means there is some w' $\sqsubseteq$ w such that $\mathcal{M}, w' \models \phi$. By applying the tableau rule R for the $\langle D\rangle$ modality, we obtain $@_s\langle D\rangle t \in \Sigma^+$ and $@_t\phi \in \Sigma^+$. Let $\mathcal{M}'$ be the same as $\mathcal{M}$ except that $w' \in V(t)$. Then we have $\mathcal{M}', w \models \langle D\rangle t$ and also, $\mathcal{M}, w' \models \phi$. ∎

Theorem 4.4 (Soundness)
If a formula ϕ is provable, then it is valid.

Proof. Suppose ϕ is provable, then, by Definition 4.1, there is a closed tableau whose root is $\neg @_i\phi$. Suppose, for proof by contradiction, that ϕ is not valid. Then, by Definition 4.2, there is a standard interval model $\mathcal{M}$ such that, for some w, we have $\mathcal{M}, w \not\models \phi$. Let $\Sigma_0 = \{\neg @_i\phi\}$, then Σ_0 is satisfied by label in $\mathcal{M}$. Suppose Σ_n is satisfied by label, we will show that Σ_{n+1} is satisfied by label. By Lemma 4.3, at least one of the sets obtainable by expanding Σ_n by R, for any R, is satisfiable by label. Let Σ_{n+1} be such a set, then Σ_{n+1} is satisfied by label. Therefore, there is an open branch of the tableau containing $\neg @_i \phi$ at its root, all of whose formulas are satisfiable by label. But this is impossible as ϕ is provable. ∎

Thus our tableau rules cannot lead us astray. If the formula $@_s \phi$ at the root of the tree is satisfiable, then trivially it is satisfiable by label. But then the proceeding lemma ensures that all formulas on at least one branch of the tableau will be satisfiable by label too.

It is time to turn to *systematic* tableau construction. We shall define a notion of systematicity general enough to prove *strong* completeness for countable languages (languages in which both Φ and Ω are countable sets). That is, ultimately we want to show that any consistent set of formulas in a countable language has a model, not just any finite set of formulas.

Now, the basic idea should be clear. Suppose Φ and Ω are both countable, and let Σ be a set of formulas in the basic hybrid language over these sets. We should pick a nominal i that does not occur in any of these formulas, prefix each of these formulas by $@_i$ and start applying the tableau rules. (We prefix by $@_i$ rather than $\neg @_i$ because we are thinking in terms of consistency rather than provability).

But there's a problem: maybe every nominal in Ω already occurs somewhere in Σ. And anyway, when we apply existential rules, we shall need to have a supply of new nominals at our disposal. So let's take care of this right away. Let NEWNOM be a countable set that is pairwise disjoint with Φ and Ω. Assume that NEWNOM has been enumerated. We shall use its elements as the new nominals needed in the systematic construction. Suppose that i is the first nominal in the enumeration of NEWNOM. Let $\Sigma^i = \{@_i \sigma \mid \sigma \in \Sigma\}$. Enumerate the elements of Σ^i and proceed as follows:

stage 1. Draw a tree consisting of a single node decorated by the first element of Σ^i. Call this T_1. Obviously T_1 is a finite tree.

stage $n + 1$. Let T_n be the finite tree constructed at stage n. We now apply all rules that are applicable to formulas (or pairs of formulas) in T_n. As T_n contains only finitely many nodes, only finitely many such applications are possible, hence as no rule returns more than two formulas, the result will be a new finite tree. (As for Ref and Ref($\sqsubseteq$), we'll assume that we apply them once for each distinct nominal on each branch.) Now, we don't really care in which order the rules are applied, but the following stipulations are important:

1. When we apply a non-branching rule, we add the formulas output by the end of every branch containing the input formula (or pair of input formulas).
2. When we apply a branching rule, we split the end of every branch containing the input formula, and add one possible output to one branch, and the other possible output to the other.
3. When we apply an existential rule, we always use the next unused nominal in NEWNOM as the new nominal we require. (There will always be such a nominal, for we can only have used up finitely many at stage n, and NEWNOM is infinite.)

Once we have applied all rules, add the $n{+}1$-th element of Σ^i to the end of every branch. Call the resulting tree T_{n+1}. Clearly T_{n+1} is finite.

The result of this process is an ω-sequence of finite trees, each of which is isomorphically in all its successors. Let T be the tree obtained as the limit of this sequence; T will embody all the information in its predecessors, and will enable us to prove strong completeness in the following section.

Figure 4 shows an example of a proof in the systematic construction establishing the validity of the formula $@_k(@_s(\langle D\rangle i \wedge \phi) \to @_i(\langle U\rangle \phi \wedge \langle U\rangle s))$ in minimal interval structures.

5 Hintikka sets and completeness

In this section we will give the completeness proof by constructing Hintikka *sets*. Let $\xi \in \{<, >, \sqsubseteq, \sqsupseteq\}$, we will write R_H for $R_{H(\xi)}$ and $\Diamond$ for $\langle\xi\rangle$.

Definition 5.1 (Hintikka sets)
A set of satisfaction statements H is called a *Hintikka set* iff it satisfies the following conditions:

1. For all atoms α, and all nominals s, if $@_s\alpha \in H$ then $\neg @_s\alpha \notin H$.
2. For all nominals s and t, if $@_s\Diamond t \in H$ then $\neg @_s\Diamond t \notin H$.
3. If a nominal s occurs in any formula in H, then $@_s s \in H$.
4. If H contains a formula that one of the branching rules can be applied to, then it contains at least one of the formulas obtainable by making this application.
5. If H contains a pair of formulas that one of the binary rules can be applied to, then it contains all the formulas obtainable by making this application.
6. If H contains a formula that one of the existential rules can be applied to, then for some nominal i it also contains the formulas that would be obtained by applying that rule to that formula using i as the new nominal a. (We shall call such a nominal i a *witness*).
7. For any other rule, if H contains a formula that one of the rules applies to, then it contains all the formulas obtainable by making this application.

Items 1, 2 and 3, which regulate what happens to atomic or near-atomic formulas, are among the most crucial demands in the definition: essentially they allow us to fix the diagram of the model we shall eventually build.

Lemma 5.2
If ϕ is consistent, then there is a Hintikka set containing $@_i\phi$.

Proof. By Definition 4.1, a formula ϕ is consistent if $\neg\phi$ is *not* provable, i.e., if there is no closed tableau with $\neg @_i \neg\phi$ at its root (for i a new nominal). In order to show that there exists a Hintikka set containing $@_i\phi$, we do the following: form a tree with $\neg @_i \neg\phi$ at its root, and apply the $\neg\neg$-rule to obtain $@_i\phi$. Now we carry out the systematic tableau construction described in the previous section. Let T be the tree obtained by this process, as ϕ is consistent, T must contain at least one open branch B. Let $\mathbf{B}$ be the set of formulas on B. It is straightforward to check that each of the 7 conditions outlined in the definition of a Hintikka set is satisfied by $\mathbf{B}$. Thus, we conclude that $\mathbf{B}$ is a Hintikka set containing $@_i\phi$. ∎

Definition 5.3
Let H be a Hintikka set. Define $Nom(H)$ to be

$$\{i |\ i \text{ is a nominal that occurs in some formula in } H\},$$

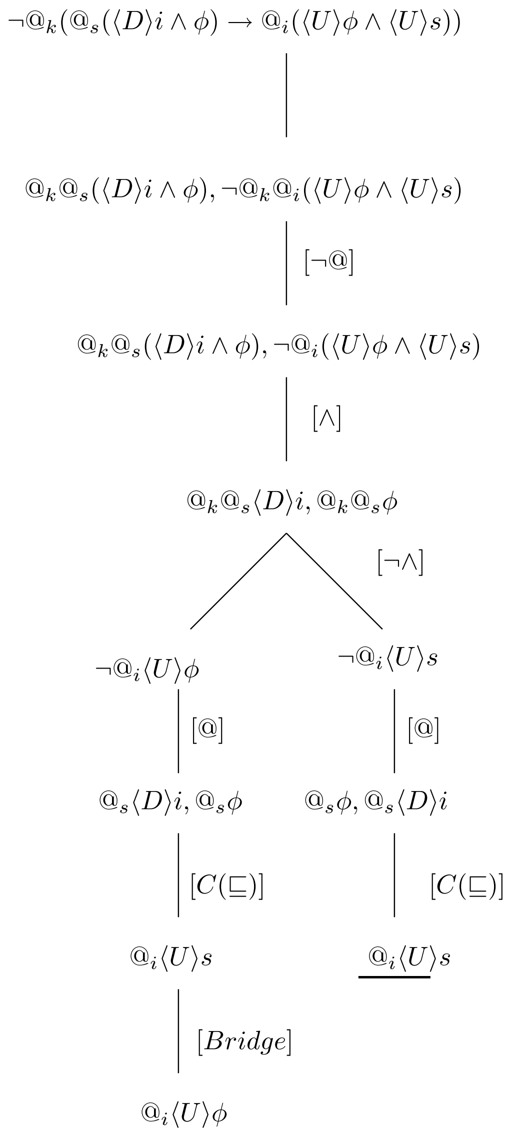

FIG. 2: An example of a systematic tableau construction. We want to show that $@_s(\langle D\rangle i \wedge \phi) \rightarrow @_i(\langle U\rangle\phi \wedge \langle U\rangle s)$ is provable. To this end, we show that there is a closed tableau whose root node is $\neg @_k(@_s(\langle D\rangle i \wedge \phi) \rightarrow @_i(\langle U\rangle\phi \wedge \langle U\rangle s))$. First, we transform this formula to its equivalent form $@_k @_s(\langle D\rangle i \wedge \phi) \wedge \neg @_k @_i(\langle U\rangle\phi \wedge \langle U\rangle s)$ (the details of which are omitted owing to lack of space). We then apply the rules systematically to obtain new formulas, and close a branch when a contradiction arises (a closed branch is indicated by an underscore at the bottom of the tree).

and define a binary relation $\sim_H$ on $Nom(H)$ by $i \sim_H j$ iff $@_i j \in H$. Clearly $\sim_H$ is an equivalence relation: item 3 in the definition of Hintikka sets ensures reflexivity, while closure under Sym and Nom guarantees symmetry and transitivity. If $k \in Nom(H)$, then $\mid k \mid$ is the equivalence class of k under $\sim_H$.

Lemma 5.4
Let H be a Hintikka set, and suppose that i,k $\in$ Nom(H). Then the following assertions are equivalent:

1. i $\sim_H$ k.
2. For every formula ϕ, $@_i\phi \in H$ iff $@_k\phi \in H$

Proof. First suppose that $i \sim_H k$. Then $@_i k \in H$, and hence $@_k i \in H$ too. But then item 2 holds because Hintikka sets are closed under Nom.

For the converse, suppose that $@_i\phi \in H$ iff $@_k\phi \in H$. As k occurs in H, we have that $@_k k \in H$.Hence, taking ϕ to be k, it follows that $@_i k \in H$, which means that $i \sim_H k$. ∎

Definition 5.5 (Induced models)
Given a Hintikka set H, let
$\mathcal{M}_H = (W_H, R_{H(<)}, R_{H(>)}, R_{H(\sqsubseteq)}, R_{H(\sqsupseteq)}, V_H)$ be any model that satisfies the following condition:

1. $W_H = \{\mid k \mid \ \mid k \in Nom(H)\}$.
2. $\mid i \mid R_H \mid j \mid$ iff $@_i\Diamond j \in H$.
3. $V_H(\alpha) = \{\mid k \mid \ \mid @_k\alpha \in H\}$, for all atoms α that occur in H. For any propositional variable p not occurring in H, $V_H(p)$ can be any subset of W_H, while for any nominal i not occurring in H, $V_H(i)$ can be any singleton subset of W_H.

To show that R_H is well-defined, we need to show that if $i \sim_H k$ and $j \sim_H l$, then $@_i\Diamond j \in H$ implies $@_k\Diamond l \in H$. So suppose $i \sim_H k$ and $j \sim_H l$. By definition this means that $@_i\Diamond j \in H$, hence as $i \sim_H k$ by Lemma 5.4 we have that $@_k\Diamond j \in H$. But as $j \sim_H l$, we also have that $@_j l \in H$. Hence, by Bridge, $@_k\Diamond l \in H$ and thus $\mid k \mid R_H \mid l \mid$ as required.

It is an immediate consequence of Lemma 5.4 that V_H is well-defined. But we also need to show that V_H is a *standard* valuation: that is, for all nominals i, $V_H(i)$ is a singleton. By definition this holds for any nominal *not* occurring in H, so suppose i occurs in H. Then $V_H(i)$ contains $\mid i \mid$, for by item 3 in the definition of Hintikka sets, $@_i i \in H$. So suppose $\mid j \mid \in V_H(i)$. But this means that $@_j i \in H$, which means that $j \sim_H i$, which means that $\mid j \mid = \mid i \mid$. Thus, for all nominals i, $V_H(i)$ is a singleton subset of W_H as required.

Then any model $\mathcal{M}_H = (W_H, R_{H(<)}, R_{H(>)}, R_{H(\sqsubseteq)}, R_{H(\sqsupseteq)}, V_H)$ of the kind just described is called a *standard interval model induced by H.*

Lemma 5.6
Let H be a Hintikka set and $\mathcal{M}_H$ a standard interval model induced by H. Then:

1. If $@_i\phi \in H$, then $\mathcal{M}_H, \mid i \mid \ \models \phi$.
2. If $\neg @_i\phi \in H$, then $\mathcal{M}_H, \mid i \mid \ \not\models \phi$.

That is, every formula in H is satisfied by label in $\mathcal{M}_H$.

PROOF. By induction on the number of connectives in ϕ. If ϕ is an atomic formula the result is clear. In the case of $\wedge$ and $\neg\wedge$, and @ and $\neg$@, it can be easily verified that items 1 and 2 hold. So suppose ϕ has the form $\Diamond j$, for some nominal j. If $@_i\Diamond j \in H$, then by the definition of R_H we have $\mid i \mid R_H \mid j \mid$. But by the case for atomic formulas, $\mathcal{M}_H, \mid j \mid \models j$, hence $\mathcal{M}_H, \mid i \mid \models \Diamond j$, as required. On the other hand, suppose that $\neg @_i\Diamond j \in H$. By item 2 in the definition of Hintikka sets this means that $@_i\Diamond j \notin H$, which means that it is *not* the case that $\mid i \mid R_H \mid j \mid$. Now, by the atomic case we know that $\mathcal{M}_H, \mid j \mid \models j$. Moreover, because V_H is a *standard* valuation we know that $\mid j \mid$ is the *only* state where j is true. Hence $\mathcal{M}_H, \mid i \mid \not\models \Diamond j$.

Next, suppose that ϕ has the form $\Diamond\psi$. We have just proved the result for the case when ψ is a nominal, so suppose that ψ is some other kind of formula. If $@_i\Diamond\psi \in H$, then as ψ is not a nominal it can be used as the input to the $\Diamond$ rule, and so by item 6 in the definition of Hintikka sets there is some witness j such that $@_i\Diamond j \in H$ and $@_j\psi \in H$. As $@_i\Diamond j \in H$, we have that $\mid i \mid R_H \mid j \mid$; and as $@_j\psi \in H$, we have that $\mathcal{M}_H, \mid j \mid \models \psi$ by the inductive hypothesis. Hence $\mathcal{M}_H, \mid i \mid \models \Diamond\psi$ as required.

Now, suppose instead that $\neg @_i\Diamond\psi \in H$. We need to show that for all $\mid k \mid$ such that $\mid i \mid R_H \mid k \mid$, $\mathcal{M}_H, \mid k \mid \not\models \psi$. So suppose that $\mid i \mid R_H \mid k \mid$; that is, $@_i\Diamond k \in H$. Applying the binary rule $\neg\Diamond$ to $\neg @_i\Diamond\psi$ and $@_i\Diamond k$ yields $\neg @_k\psi$, hence as H is a Hintikka set, by item 5, $\neg @_k\psi \in H$. By the inductive step for satisfaction statements we thus have that $\mathcal{M}_H$, $\mid k \mid \not\models \psi$, hence as $\mid k \mid$ was an arbitrary successor of $\mid i \mid$ it follows that $\mathcal{M}_H, \mid i \mid \not\models \Diamond\psi$ as required.

∎

LEMMA 5.7
Let $\mathcal{M}_H = (W_H, R_{H(<)}, R_{H(>)}, R_{H(\sqsubseteq)}, R_{H(\sqsupseteq)}, V_H)$ be a standard interval model induced by H. Then $(W_H, R_{H(<)}, R_{H(>)}, R_{H(\sqsubseteq)}, R_{H(\sqsupseteq)})$ is a minimal interval structure.

PROOF. That $(W_H, R_{H(<)}, R_{H(>)}, R_{H(\sqsubseteq)}, R_{H(\sqsupseteq)})$ is a minimal interval structure follows from the general completeness result in [7]. For illustrative reasons, we will show that $<$ and $\sqsupseteq$ preserve *monotonicity*. We show this for right monotonicity. Suppose, for proof by contradiction, that for some w_1, w_2, $w_3 \in W_H$, we have $w_1 \sqsupseteq w_2$, $w_1 < w_3$, and $w_2 \not< w_3$. Let $w_1 \in V_H(i)$, $w_2 \in V_H(j)$, $w_3 \in V_H(k)$, then we have $@_i\langle D\rangle j \in H$, $@_i\langle F\rangle k \in H$, and $@_j\langle F\rangle k \notin H$, and therefore $\mathcal{M}_H \models @_i\langle D\rangle j$, $\mathcal{M}_H \models @_i\langle F\rangle k$ and $\mathcal{M}_H \not\models @_j\langle F\rangle k$. However, we can apply the MON rule on $@_i\langle D\rangle j$ and $@_i\langle F\rangle k$ so as to obtain $@_j\langle F\rangle k \in H$, and therefore $\mathcal{M}_H \models @_j\langle F\rangle k$.

∎

THEOREM 5.8 (Completeness)
If a formula ϕ is consistent, then there is a model of ϕ based on a minimal interval structure.

PROOF. If ϕ is consistent, by lemma 5.2, we obtain a Hintikka set H containing $@_i\phi$. Let $\mathcal{M}_H$ be a standard interval model induced by H, then $\mathcal{M}_H \models @_i\phi$ by lemma 5.6. By the preceding lemma, $\mathcal{M}_H$ is based on a minimal interval structure. ∎

5.1 Minimal interval structures with conjectivity

By applying the method for *node-creating rules* given in [7], we can extend the tableau calculus of section 3 to incorporate an extra rule: the Conjectivity Rule (*CONJ*), so as to capture the basis of van Benthem's Minimal Interval Structure. However, this is a more complicated rule involving two parts: the first, the *create glb*-rule, creates a maximum sub-interval $l_{s,u}$, or the *greatest lower bound* (*glb*), for any two overlapping intervals s and u - this rule is intended to be applied only once (akin to the Ref- and Ref($\sqsubseteq$)-rules; the second, the *glb constraint*-rule, takes any 2 overlapping intervals s and u for which a *glb* has been created and forces every common subinterval t to be included in the *glb* $l_{s,u}$. To put it schematically, the *CONJ* Rule incorporates the following:

$$\frac{@_s\langle D\rangle t, @_t\langle U\rangle u}{@_s\langle D\rangle l_{s,u}, @_{l s,u}\langle U\rangle u}[\textit{create glb}] \qquad \frac{@_s\langle D\rangle t, @_t\langle U\rangle u, \text{[create glb]}}{@_t\langle U\rangle l_{s,u}}[\textit{glb constraint}]$$

This is sound. For, if Σ is a set of satisfaction statements that is satisfiable by label, then the set Σ^+, obtained from Σ by an application of the *CONJ* rule is also satisfiable by label. Suppose $@_s\langle D\rangle t$ and $@_t\langle U\rangle u$ are in Σ. This means there is a standard model $\mathcal{M}$ such that $\mathcal{M} \models @_s\langle D\rangle t \wedge @_t\langle U\rangle u$. For $w_0, w_1, w_2 \in \mathcal{M}$, let $w_0 \in V(s)$, $w_1 \in V(t)$ and $w_2 \in V(u)$. By applying the first part of the CONJ rule, we obtain $@_s\langle D\rangle l \in \Sigma^+$ and $@_l\langle U\rangle u \in \Sigma^+$. Let $\mathcal{M}'$ be the same as $\mathcal{M}$ except for some $w' \in \mathcal{M}$ such that $w_0 \sqsupseteq w' \sqsubseteq w_2$ and w' is the largest sub-interval contained in w_0 and w_2, we have $w' \in V(l)$ (where w' is not necessarily distinct from w_1). Now by applying the second part of the CONJ rule, we obtain $@_t\langle U\rangle l \in (\Sigma^+)^+$. So we have $w_1 \sqsubseteq w'$ and $\mathcal{M}', w_1 \models \langle U\rangle l$.

The systematic construction of section 4 goes through without any difficulty, and consequently we can establish the necessary completeness result.

Lemma 5.9
Let H be a Hintikka set with respect to van Benthem's Minimal Interval Struture (incorporating the CONJ rule). Let
$\mathcal{M}_H = (W_H, R_{H(<)}, R_{H(>)}, R_{H(\sqsubseteq)}, R_{H(\sqsupseteq)}, V_H)$ be a standard interval model induced by H. Then $(W_H, R_{H(<)}, R_{H(>)}, R_{H(\sqsubseteq)}, R_{H(\sqsupseteq)})$ is a van Bentham Minimal Interval Structure.

Proof. We only need to show that $(W_H, R_{H(<)}, R_{H(>)}, R_{H(\sqsubseteq)}, R_{H(\sqsupseteq)})$ satisfies conjectivity. So , suppose that for some w_1, w_2, and $w_3 \in W_H$, we have $w_1 \sqsupseteq w_2$ and $w_2 \sqsubseteq w_3$. Let $w_1 \in V_H(i)$, $w_3 \in V_H(j)$, and $w_2 \in V_H(k)$, then we have $@_i\langle D\rangle k \in H$, and $@_k\langle U\rangle j \in H$, and therefore $\mathcal{M}_H \models @_i\langle D\rangle k$ and $\mathcal{M}_H \models @_k\langle U\rangle j$. Now we can apply the first step in the CONJ rule to create a *greatest lower bound* (*glb*) $= l$ such that $@_i\langle D\rangle l \in H$ and $@_l\langle U\rangle j \in H$. Now for any w_4 such that $w_1 \sqsupseteq w_4 \sqsubseteq w_3$, we have, by the second part of the CONJ rule, that $@_m\langle U\rangle l$ (where $w_4 \in V_H(m)$) and therefore $\mathcal{M}_H \models @_m\langle U\rangle l$. ∎

Theorem 5.10 (Completeness for van Benthem's minimal interval structures)
If ϕ is consistent, then there is a model of ϕ based on van Benthem's minimal interval structure.

Proof. Identical to the proof of Theorem 5.8. ∎

Example 5.11
The addition of the conjectivity condition suggests that the logic of van Benthem minimal interval structures is different from the logic of minimal interval structures. And indeed this is the case as can be shown by the example in Figure 3. Let $\theta = \langle D\rangle(p \wedge \langle U\rangle i) \wedge \langle D\rangle(\neg p \wedge \langle U\rangle i) \rightarrow \langle D\rangle(\langle D\rangle p \wedge \langle D\rangle\neg p \wedge \langle U\rangle i)$ (where p is a propositional variable, and i is a nominal). Then θ is not true at interval A in the above minimal interval structure. However, it can be checked that θ is a valid formula of a van Benthem minimal interval structure. Therefore, the logic of van Benthem minimal interval structures is different from the logic of minimal interval structures.

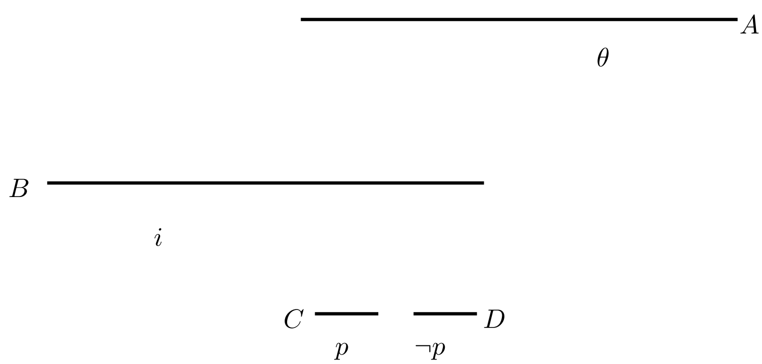

FIG. 3: Above we have a minimal interval structure consisting of 4 intervals A, B, C and D. The relations between the intervals are depicted pictorially, so for example C and D are included in both A and B, and C precedes D. Likewise, the formulae true at each interval is inserted below, so for instance, $\neg p$ holds at interval D.

6 Finite models for pre-interval structures

In this section we will prove a general truth lemma stating that, for any formula ϕ, if ϕ is satisfiable in a model based on a minimal interval structure, then ϕ is satisfiable in a finite model based on a pre-interval structure.

6.1 Labelling and finite models

Let ϕ be a formula. Let n denote $\mid \phi \mid$. Take Φ_ϕ to be the smallest set of well-formed formulae (*wff*) closed under single negations containing the subformulae of ϕ such that if $[F]\beta \in \Phi_\phi$, then both $[D]\beta \in \Phi_\phi$ and $[U][F]\beta \in \Phi_\phi$, *and*, if $[P]\beta \in \Phi_\phi$, then both $[D]\beta \in \Phi_\phi$ and $[U][P]\beta \in \Phi_\phi$. And furthermore, for $\beta \neq [F]\gamma$, if $[F]\beta \in \Phi_\phi$ then $[F][F]\beta \in \Phi_\phi$. Similarly, for $\beta \neq [P]\gamma$, if $[P]\beta \in \Phi_\phi$ then $[P][P]\beta \in \Phi_\phi$. Φ_ϕ is finite and $\phi \in \Phi_\phi$. It can be checked that the size of Φ_ϕ is linear in the size of n.

DEFINITION 6.1
We call $S \subseteq \Phi_\phi$ a *nice* set if it satisfies the following properties:

1. S is maximally propositionally consistent,
2. if $[U]\beta \in S$ then $\beta \in S$,
3. if $[D]\beta \in S$ then $\beta \in S$,
4. if $[F]\beta \in S$ then $[U][F]\beta \in S$,
5. if $[P]\beta \in S$, then $[U][P]\beta \in S$,
6. if $[F]\beta \in S$ and $[F][F]\beta \in \Phi_\phi$, then $[F][F]\beta \in S$,
7. if $[P]\beta \in S$ and $[P][P]\beta \in \Phi_\phi$, then $[P][P]\beta \in S$.

Given a set $\mathfrak{N}$ of nice sets, we define the relations $R^\dagger_<$, $R^\dagger_>$, $R^\dagger_\sqsubseteq$, $R^\dagger_\sqsupseteq$ as follows:

1. For any $S, S' \in \mathfrak{N}$, $SR^\dagger_< S'$ if and only if, for every *wff* $[F]\beta \in \Phi_\phi$ and every *wff*

$[P]\gamma \in \Phi_\phi$, if $[F]\beta \in S$, then $[F]\beta \in S'$, $[D]\beta \in S'$ and $\beta \in S'$ *and* if $[P]\gamma \in S'$, then $[P]\gamma \in S$, $[D]\gamma \in S$ and $\gamma \in S$, *and* we define $R^\dagger_>$ as the converse.
2. For any $S, S' \in \mathfrak{N}$, $SR^\dagger_\sqsupseteq S'$ if and only if, for every *wff* $[D]\beta \in \Phi_\phi$ and every *wff* $[U]\gamma \in \Phi_\phi$, if $[D]\beta \in S$, then $[D]\beta \in S'$ *and* if $[U]\gamma \in S'$, then $[U]\gamma \in S$, *and* again $R^\dagger_\sqsupseteq$ is defined as the converse.

It is straightforward to check that $R^\dagger_\sqsubseteq$ is reflexive and both $R^\dagger_<$ and $R^\dagger_\sqsubseteq$ are transitive. We will show that the definitions of $R^\dagger_<$ and $R^\dagger_\sqsubseteq$ satisfy the left and right *Monotonicity* condition. For the case of right monotonicity, suppose we have that $S_0R^\dagger_\sqsubseteq S_1$, and $S_1R^\dagger_< S_2$. We want to show that $S_0R^\dagger_< S_2$. Assume that $[F]\beta \in S_0$ (for $[F]\beta \in \Phi_\phi$), then $[U][F]\beta \in S_0$. Now, since $S_0R^\dagger_\sqsubseteq S_1$, we have that $[U][F]\beta \in S_1$ and $[U][F]\beta \in \Phi_\phi$. By the reflexive closure of S_1, we obtain $[F]\beta \in S_1$. Since, $S_1R^\dagger_< S_2$, we have that $[F]\beta \in S_2$, $\beta \in S_2$ and $[D]\beta \in S_2$. Now suppose $[P]\gamma \in S_2$ (for $[P]\gamma \in \Phi_\phi$). By $S_1R^\dagger_< S_2$ we have $[P]\gamma \in S_1$, $[D]\gamma \in S_1$ and $\gamma \in S_1$. Since $S_0R^\dagger_\sqsubseteq S_1$, we obtain $[D]\gamma \in S_0$. And by the reflexive closure of S_0, we have $\gamma \in S_0$. So, it remains to show that $[P]\gamma \in S_0$. Now, since $[P]\gamma \in S_2$, we have two possible cases to consider. First, suppose $\gamma \neq [P]\alpha$ (for any α), then $[P][P]\gamma \in \Phi_\phi$ and $[P][P]\gamma \in S_2$. Since $S_1R^\dagger_< S_2$, we have $[P][P]\gamma \in S_1$, $[P]\gamma \in S_1$ and $[D][P]\gamma \in S_1$. Given that $S_0R^\dagger_\sqsubseteq S_1$, we obtain $[D][P]\gamma \in S_0$ and by the reflexive closure of S_0 we have $[P]\gamma \in S_0$, as was required. For the second case, suppose $\gamma = [P]\alpha$, then $[P][P]\alpha \in S_2$. Since $S_1R^\dagger_< S_2$, we have $[P][P]\alpha \in S_1$, $[P]\alpha \in S_1$ and $[D][P]\alpha \in S_1$. Given that $S_0R^\dagger_\sqsubseteq S_1$, we obtain $[D][P]\alpha \in S_0$ and by the reflexive closure of S_0 we have $[P]\alpha \in S_0$. Since $[P][P]\alpha \in \Phi_\phi$, we have $[P][P]\alpha \in S_0$, and therefore $[P]\beta \in S_0$, as was required. We conclude that $S_0R^\dagger_< S_2$ and right monotonicity is established. Analogous reasoning establishes that left monotonicity is also satisfied.

$\mathfrak{N}$ is ϕ-*saturated* if it satisfies the following conditions:

1. $\exists S \in \mathfrak{N}$ with $\phi \in S$.
2. *If* $S \in \mathfrak{N}$ and $\Box\psi \in \Phi_\phi$ and $\Box\psi \notin S$ (for $\Box \in \{[F],[P],[U],[D]\}$), *then* $\exists S' \in \mathfrak{N}$ with (1) $\psi \notin S'$ and (2) $SR^\dagger_\theta S'$ (for $\theta \in \{<,>,\sqsubseteq,\sqsupseteq\}$ corresponding to $\Box$).
3. For any nominal $i \in \Phi_\phi$, there is a unique $S^i \in \mathfrak{N}$ with $i \in S^i$.
4. For all $@_i\psi \in \Phi_\phi$, (a) $@_i\psi \in S$ for some $S \in \mathfrak{N} \Rightarrow @_i\psi \in S$ for all $S \in \mathfrak{N}$;
   (b) $\psi \in S^i \iff @_i\psi \in S$ for all $S \in \mathfrak{N}$.

Lemma 6.2
Suppose ϕ is satisfiable in a model based on a minimal interval structure, then there is a ϕ-saturated set $\mathfrak{N}$.

Proof. Suppose ϕ is satisfiable in some model $\mathcal{M} = (W, R_<, R_>, R_\sqsubseteq, R_\sqsupseteq, V)$ based on a minimal interval structure. For $w \in W$, let $S_w = \{\psi : \psi \in \Phi_\phi, \mathcal{M}, w \models \psi\}$. Let $\mathfrak{N} = \{S_w : w \in \mathcal{M}\}$. Then $\mathfrak{N}$ is ϕ-saturated. ■

Lemma 6.3 (Truth Lemma)
If $\mathfrak{N}$ is ϕ-saturated, then there is a model $\mathcal{M} = (\mathfrak{N}, R^\dagger_<, R^\dagger_>, R^\dagger_\sqsubseteq, R^\dagger_\sqsupseteq, V)$ such that $\forall\psi \in \Phi_\phi$ and $\forall S \in \mathfrak{N}$,

$$\mathcal{M}, S \models \psi \iff \psi \in S.$$

Hence, $\mathcal{M}, S \models \phi$ for some $S \in \mathfrak{N}$

Proof. We will only use conditions 2-4 of the definition of ϕ-saturated. We define the valuation as follows: let $ATOM$ be the set of propositional variables occurring in Φ_ϕ and define $V : ATOM \to \mathcal{P}(\mathfrak{N})$ by $V(p) = \{S \in \mathfrak{N} : p \in S\}$. It remains to prove the equivalence. This is done by induction on the complexity of ψ. The atomic case holds by definition of V and the Boolean cases are straightforward. So, suppose $\psi = [D]\alpha$. For the left to right direction, suppose $\mathcal{M}, S \models [D]\alpha$. We want to show that $[D]\alpha \in S$. Suppose, for proof by contradiction, that $[D]\alpha \notin S$. As $\mathfrak{N}$ is ϕ-saturated, this means there is a S' with $SR^\dagger_\sqsupseteq S'$ and $\alpha \notin S'$. By the inductive hypothesis, we have $\mathcal{M}, S' \not\models \alpha$, and therefore $\mathcal{M}, S \not\models [D]\alpha$. We thus conclude that $[D]\alpha \in S$. For the right to left direction, suppose $[D]\alpha \in S$. We want to show that $\mathcal{M}, S \models [D]\alpha$. Suppose $SR^\dagger_\sqsupseteq S'$. From $[D]\alpha \in S$, we have by definition of $R^\dagger_\sqsupseteq$ that $[D]\alpha \in S'$. By the reflexive closure, we have $\alpha \in S'$. By the inductive hypothesis, we have $\mathcal{M}, S' \models \alpha$ and therefore $\mathcal{M}, S \models [D]\alpha$. The case of the other modalities is analogous. Now suppose $\psi = @_i\alpha$. For the left to right direction, suppose $\mathcal{M}, S \models @_i\alpha$. We want to show that $@_i\alpha \in S$. Suppose, for proof by contradiction, that $@_i\alpha \notin S$. As $\mathfrak{N}$ is ϕ-saturated, we have $\alpha \notin S^i$ and $@_i\alpha \notin S$ for all $S \in \mathfrak{N}$. By the inductive hypothesis, we have $\mathcal{M}, S^i \not\models \alpha$, and therefore $\mathcal{M}, S \not\models @_i\alpha$. For the right to left direction, suppose $@_i\alpha \in S$. We want to show that $\mathcal{M}, S \models @_i\alpha$. From $@_i\alpha \in S$ we have $@_i\alpha \in S$ for all $S \in \mathfrak{N}$ and $\alpha \in S^i$. By the inductive hypothesis, we have $\mathcal{M}, S^i \models \alpha$, and therefore $\mathcal{M}, S \models @_i\alpha$. ∎

In particular, we conclude that ϕ is satisfiable if and only if there is a ϕ-saturated set.

Lemma 6.4
If ϕ is satisfiable in a model based on a minimal interval structure, then ϕ has a model based on a pre-interval structure of exponential size.

Proof. Suppose ϕ is satisfiable. By Lemma 6.2, we have that there is a ϕ-saturated set $\mathfrak{N}$. By Lemma 6.3, $\mathcal{M} = (\mathfrak{N}, R^\dagger_<, R^\dagger_>, R^\dagger_\sqsubseteq, R^\dagger_\sqsupseteq, V)$ is a model in which ϕ is satisfied. And $\mathfrak{N} \subseteq \mathcal{P}(\Phi_\phi)$. So $| \mathfrak{N} | \leq 2^{|\Phi_\phi|} \leq 2^{an}$. ∎

7 Bulldozing for minimal interval structures

In this section, we will show that the logic of minimal interval structures is decidable. From section 6.1 we have that, for any formula ϕ, if ϕ is satisfiable in a model $\mathcal{A}$ based on a minimal interval structure, then ϕ is satisfiable in a finite model $\mathcal{B}$ based on a pre-interval structure. The model thus obtained will not in general be a minimal interval structure; however, we will show in this section that by bulldozing we can obtain a model $\mathcal{C}$ based on a minimal interval structure from *any* model $\mathcal{B}$ (finite or otherwise) based on a pre-interval structure. As a consequence we obtain the decidability of the logic of minimal interval structures.

In order to obtain the model $\mathcal{C}$ we need some extra properties to handle nominals. The extra properties are enforced by the satisfiability of ϕ^+ as follows:

Definition 7.1
Define ϕ^+ to consist of the conjunction of the following:

1. ϕ
2. $\bigwedge\{@_i[F]\neg i : i \text{ occurring in } \phi\}$;
3. $\bigwedge\{@_i[D](\neg i \to [D]\neg i) : i \text{ occurring in } \phi\}$;
4. $\bigwedge\{@_i\neg\langle P\rangle\langle U\rangle i : i \text{ occurring in } \phi\}$;
5. $\bigwedge\{@_i\neg\langle F\rangle\langle U\rangle i : i \text{ occurring in } \phi\}$

Obviously, if ϕ does not contain any nominals then $\phi^+ = \phi$. Let n denote $\mid \phi^+ \mid$. We then take Φ_{ϕ^+} with respect to ϕ^+. It can be checked that the size of Φ_{ϕ^+} is $\leq bn$ for some fixed b.

Now suppose ϕ is satisfiable in a minimal interval structure. Then it follows immediately that ϕ^+ is satisfiable in a minimal interval structure. By Lemma 6.2, we have there exists a ϕ^+-saturated set $\mathfrak{N}$ with $\phi^+ \in S \in \mathfrak{N}$ (for some S). And by Lemma 6.3 there is a finite model $\mathcal{B}$ of ϕ^+ in which $<$ is transitive, $\sqsubseteq$ is reflexive and transitive, and both relations satisfy monotonicity. We will now show that if ϕ^+ is satisfiable in $\mathcal{B}$, then ϕ^+ (and hence ϕ) is satisfiable in a model based on a minimal interval structure.

Let $\mathcal{M} = (W, R_<, R_>, R_\sqsubseteq, R_\sqsupseteq, V)$ be a model for ϕ^+ based on a pre-interval structure. $\mathcal{M}$ may be finite or infinite. We know that $\mathcal{M}$ may contain $<$-clusters and $\sqsubseteq$-clusters, which we will bulldoze away. We define maximal $\sqsubseteq$-clusters and maximal $<$-clusters. Let $\approx_\sqsubseteq$ and $\approx_<$ be defined as follows:

$x \approx_\sqsubseteq y$ iff $(xR_\sqsubseteq y \wedge yR_\sqsubseteq x)$, and
$x \approx_< y$ iff $(xR_< y \wedge yR_< x)$

Then $R_\sqsubseteq$ defines an equivalence relation over W, and $R_<$ defines an equivalence relation over the set $\{w \in W : wR_< w\}$ (possibly empty). A cluster will be an equivalence class of one of these forms.

If a cluster consists of only a single reflexive point then it is called a *simple* cluster, otherwise it is a *proper* cluster.

From now on any reference to nominals will refer to those occurring in ϕ^+.

Lemma 7.2
A nominal can only name a $<$-irreflexive world contained in a simple $\sqsubseteq$-cluster.

Proof. First, let C be a $<$-cluster and let $a \in C$. Suppose, for proof by contradiction, that for a nominal i, $V(i) = \{a\}$, then we have $aR_< a$. However, for any nominals occurring in ϕ^+, we have that $@_i \neg \langle F \rangle i$ is satisfied in $\mathcal{M}$ and so $\neg(aR_< a)$. Now we show that any world occurring in a proper $\sqsubseteq$-cluster cannot be the denotation of a nominal. Let D be a proper $\sqsubseteq$-cluster, where $b \in D$ such that $V(i) = \{b\}$. For proof by contradiction, choose any $c \in D$ such that $b \neq c$. Then we have $bR_\sqsubseteq c$ and $cR_\sqsubseteq b$. However, for any nominals occurring in ϕ^+ we have that $@_i[D](\neg i \rightarrow [D]\neg i)$ is satisfied in $\mathcal{M}$ and so $\neg(bR_\sqsubseteq c)$. From the above contradictions we conclude that a nominal can only name a $<$-irreflexive world contained in a simple $\sqsubseteq$-cluster. ■

By the previous Lemma, we know that the worlds named by nominals do not occur in any of the problematic clusters, and therefore can be handled separately. We are now ready to prove the following theorem:

Theorem 7.3
If ϕ^+ is satisfiable in $\mathcal{M}$, then ϕ^+ is satisfiable in a model based on a minimal interval structure.

Proof. Let $\mathcal{Q} = (U, <', >', \sqsubseteq', \sqsupseteq')$, where $U = \{(n,m) : n < m \text{ and } n, m \in \mathbb{Q}\}$, and $<' = \{((n,m),(k,l)) : n < m \leq k < l\}$, $\sqsubseteq' = \{((n,m),(k,l)) : k \leq n < m \leq l\}$. We know from example 2 in section 2 that $\mathcal{Q}$ is a minimal interval structure. Furthermore, let $\sqsubset' = \{((n,m),(k,l)) : k < n < m < l\}$. We define $>'$ and $\sqsupseteq'$ to be the converse of $<'$ and $\sqsubseteq'$ respectively.

Let W^- be the set of worlds that are named by nominals in $\mathcal{M}$. We will use $k, l, m \ldots$ to denote the worlds in W^-, and use $a, b, \ldots$ and $x, y, \ldots$ to denote arbitrary worlds of W. Now we will construct a product of $\mathcal{F}$ and $\mathcal{Q}$, where $\mathcal{F}$ is the frame of $\mathcal{M}$. Let $\mathcal{F} \times \mathcal{Q} = \langle (((W \backslash W^-) \times U) \cup \{(a,a) : a \in W^-\}), <^*, >^*, \sqsubseteq^*, \sqsupseteq^* \rangle$.

The $<^*$ relation is defined as:

$(a, a') <^* (b, b') \iff$
1. $a, b \notin W^- \wedge aR_< b \wedge a' <' b'$; or
2. $\exists k \exists y (aR_< kR_{\sqsupseteq} yR_{\leq} b)$; or
3. $\exists k \exists y (aR_{\leq} yR_{\sqsubseteq} kR_< b)$.

where, $xR_{\leq} y \iff xR_< y \vee x = y$, and $>^*$ is defined as the converse of $<^*$.

The $\sqsubseteq^*$ relation is defined as:

$(a, a') \sqsubseteq^* (b, b') \iff$
1. $(a, a') = (b, b')$; or
2. $(a, a') \sqsubset^* (b, b')$; or
3. $\exists k (aR_{\sqsubseteq} kR_{\sqsubseteq} b)$.

where, $(a, a') \sqsubset^* (b, b') \iff a, b \notin W^- \wedge aR_{\sqsubseteq} b \wedge a' \sqsubset' b'$, and $\sqsupseteq^*$ is defined as the converse of $\sqsubseteq^*$.

Claim 1. (i) If $(a, a') <^* (b, b')$, then $aR_< b$. (ii) If $(a, a') \sqsubseteq^* (b, b')$, then $aR_{\sqsubseteq} b$.

Proof. Proof of (i): suppose $(a, a') <^* (b, b')$. We want to show that $aR_< b$. If we have $aR_< b \wedge a' <' b'$, then it is immediate. So suppose we have $\exists k \exists y (aR_< kR_{\sqsupseteq} yR_{\leq} b)$. Then by monotonicity, we have $aR_< yR_{\leq} b$, and by transitivity, we have $aR_< b$. The remaining case is analogous. Proof of (ii): suppose we have $(a, a') \sqsubseteq^* (b, b')$. We want to show that $aR_{\sqsubseteq} b$. If we have $(a, a') = (b, b') \vee (a, a') \sqsubset^* (b, b')$, then it is immediate that $aR_{\sqsubseteq} b$. So, suppose we have $\exists k (aR_{\sqsubseteq} kR_{\sqsubseteq} b)$. Then by transitivity, we have $aR_{\sqsubseteq} b$. ∎

Now we define a valuation V^* on $\mathcal{F} \times \mathcal{Q}$ by $(x, x') \in V^*(p) \iff x \in V(p)$, for all propositional atoms p; for any nominal i, we have $(x, x) \in V^*(i) \iff x \in V(i)$ for $x \in W^-$. Let $\mathcal{M}^* = \langle \mathcal{F} \times \mathcal{Q}, V^* \rangle$.

Claim 2. The mapping f: $\mathcal{F} \times \mathcal{Q} \to \mathcal{F}$, such that $f((x, x')) = x$, is a surjective bounded morphism, and the model $\mathcal{M}$ is a bounded morphic image of $\mathcal{M}^*$ under f.

Proof. f is obviously surjective. Recall that a mapping f: $\mathcal{F} \times \mathcal{Q} \to \mathcal{F}$ is a *bounded morphism* if it satisfies the following conditions:

(i) (x, x') and $f((x, x'))$ satisfy the same proposition letters.
(ii) f is a homomorphism with respect to all the relations (for example, if $(x, x') \sqsubseteq^* (y, y')$ then $f((x, x'))R_{\sqsubseteq} f((y, y')))$.
(iii) The *back condition* for all relations (for example, if $f((x, x'))R_{\sqsubseteq} y$ then there exists (y, y') such that $(x, x') \sqsubseteq^* (y, y')$ and $f((y, y')) = y$).

First, we treat the $<^*$ relation. (i) is satisfied by definition of V^*. (ii) by Claim 1. For proof of (iii): suppose $f((a, a'))R_< b$, we want to show that there is a (b, b') such that $(a, a') <^* (b, b')$ and $f((b, b')) = b$. Now either $b \in W \backslash W^-$ and then there is some $b' \in U$ such that $a' <' b'$; or $b \in W^-$, in which case we have $\exists b(aR_< bR_\sqsupseteq b)$ and therefore $(a, a') <^* (b, b)$ and $f((b, b)) = b$. $>^*$ is treated similarly.

Now we treat the $\sqsubseteq^*$ relation. (i) is satisfied by definition of V^*. (ii) by Claim 1. For proof of (iii): suppose $f((a, a'))R_\sqsubseteq b$, we want to show that there is a (b, b') such that $(a, a') \sqsubseteq^* (b, b')$ and $f((b, b')) = b$. Now either $b \in W \backslash W^-$ and then there is some $b' \in U$ such that $(a, a') \sqsubseteq^* (b, b')$ and $f((b, b')) = b$; or $b \in W^-$, in which case we have $\exists b(aR_\sqsubseteq bR_\sqsubseteq b)$ and therefore $(a, a') \sqsubseteq^* (b, b)$ and $f((b, b)) = b$. $\sqsupseteq^*$ is treated similarly.

This concludes the proof of the claim. ∎

Claim 3. The product structure $\mathcal{F} \times \mathcal{Q}$ is a minimal interval structure.

Proof. We first check that $<^*$ is a strict ordering. Suppose $(a, a') <^* (c, c')$. We want to show that $(a, a') \neq (c, c')$. By definition of $<^*$ there are 3 possible ways in which $(a, a') <^* (c, c')$ holds. If we have $aR_< c \wedge a' <' c'$, then by the strict ordering of $<'$ we have $(a, a') \neq (c, c')$. Now suppose we have $\exists k \exists y(aR_< kR_\sqsupseteq yR_\leq c)$, and suppose for proof by contradiction that $(a, a') = (c, c')$. Then we have $aR_< kR_\sqsupseteq yR_\leq aR_< k$. By transitivity of $R_<$, we have $kR_\sqsupseteq yR_< k$. However, for any nominal i in ϕ^+ such that $V(i) = \{k\}$, we have that $@_i \neg \langle P \rangle \langle U \rangle i$ is satisfied in $\mathcal{M}$, and so $\neg(kR_\sqsupseteq yR_< k)$. Therefore $(a, a') \neq (c, c')$. The third case is analogous to the second using $@_i \neg \langle F \rangle \langle U \rangle i$. We therefore conclude that $<^*$ is irreflexive.

Now for transitivity suppose $(a, a') <^* (b, b') <^* (c, c')$. We want to show $(a, a') <^* (c, c')$. By Claim 1 we have $aR_< b$ and $bR_< c$. We have a number of possible cases to consider. Suppose we have $aR_< b \wedge a' <' b'$ and $bR_< c \wedge b' <' c'$. Then by the transitivity of both $R_<$ and $<'$, we have $aR_< c \wedge a' <' c'$, and therefore $(a, a') <^* (c, c')$. Now suppose we have $aR_< b$ and $\exists k \exists x(bR_< kR_\sqsupseteq xR_\leq c)$. Since we have $aR_< bR_< k$, by transitivity we obtain $aR_< kR_\sqsupseteq xR_\leq c$, and therefore $(a, a') <^* (c, c')$. Now suppose we have $\exists k \exists x(aR_< kR_\sqsupseteq xR_\leq b)$ and $\exists l \exists y(bR_< lR_\sqsupseteq yR_\leq c)$. By monotonicity we get $aR_< xR_\leq bR_< l$, and by transitivity we obtain $aR_< l$, and so we have $aR_< lR_\sqsupseteq yR_\leq c$, and therefore $(a, a') <^* (c, c')$. All other cases are analogous to one of the above. We therefore conclude that $<^*$ is transitive.

Now, we check that $\sqsubseteq^*$ is a partial order. First, we check reflexivity. We want to show that $(a, a') \sqsubseteq^* (a, a')$. This is immediate since the first case always holds.

For transitivity, suppose $(a, a') \sqsubseteq^* (b, b') \sqsubseteq^* (c, c')$. We want to show that $(a, a') \sqsubseteq^* (c, c')$. By Claim 1, we have $aR_\sqsubseteq b$ and $bR_\sqsubseteq c$. If either $(a, a') = (b, b')$ or $(b, b') = (c, c')$, then it follows immediately that $(a, a') \sqsubseteq^* (c, c')$. So, suppose we have $(a, a') \sqsubset^* (b, b') \sqsubset^* (c, c')$. Then by the transitivity of $R_\sqsubseteq$ and $\sqsubset'$, we have $aR_\sqsubseteq c \wedge a' \sqsubset' c'$, and therefore $(a, a') \sqsubseteq^* (c, c')$. Now suppose we have $aR_\sqsubseteq b$ and $\exists k(bR_\sqsubseteq kR_\sqsubseteq c)$. By the transitivity of $R_\sqsubseteq$ we have $aR_\sqsubseteq kR_\sqsubseteq c$, and therefore $(a, a') \sqsubseteq^* (c, c')$. The cases where we have either $\exists k(aR_\sqsubseteq kR_\sqsubseteq b)$ and $bR_\sqsubseteq c$ or $\exists k(aR_\sqsubseteq kR_\sqsubseteq b)$ and $\exists l(bR_\sqsubseteq lR_\sqsubseteq c)$ are treated analogously. We therefore conclude that $\sqsubseteq^*$ is transitive.

It remains to check that $\sqsubseteq^*$ is antisymmetric. So, suppose $(a, a') \sqsubseteq^* (b, b') \sqsubseteq^* (a, a')$. We want to show that $(a, a') = (b, b')$. Suppose, for proof by contradiction, that $(a, a') \sqsubset^* (b, b') \sqsubset^* (a, a')$. Then we have $a' \sqsubset' b' \sqsubset' a'$, and therefore $a' \sqsubset' a'$, which is impossible. All other cases involve nominals, and each one via the transitivity of $R_\sqsubseteq$ gives us $\exists k(aR_\sqsubseteq kR_\sqsubseteq a)$. By Lemma 7.2, we have that a nominal can only name a world in a $<$-irreflexive, simple $\sqsubseteq$-cluster. So, $a = k = b$. We therefore conclude that $(a, a') = (b, b')$. Thus $\sqsubseteq^*$ is antisymmetric.

Finally, we have to check that the relations are monotonous. We will treat left monotonicity. So, suppose $(a, a') \sqsubseteq^* (b, b') <^* (c, c')$. We want to show that $(a, a') <^* (c, c')$. If

$(a, a') = (b, b')$, then it is immediate that $(a, a') <^* (c, c')$. So, suppose $(a, a') \sqsubset^* (b, b')$ and $bR_< c \wedge b' <' c'$. Then we have $aR_\sqsubseteq bR_< c$ and $a' \sqsubset' b' <' c'$, and by the monotonicity of both $R_\sqsubseteq$ and $R_<$, and $\sqsubset'$ and $<'$, we obtain $aR_< c$ and $a' <' c'$, and therefore $(a, a') <^* (c, c')$. Now suppose $(a, a') \sqsubset^* (b, b')$ and $\exists k \exists y (bR_\leq y R_\sqsubseteq k R_< c)$. Since $aR_\sqsubseteq$ b, by monotonicity we have either $aR_\leq y$ or $aR_\sqsubseteq y$. In the first case we have $aR_\leq y R_\sqsubseteq k R_< c$, and therefore $(a, a') <^* (c, c')$. In the second case, by transitivity of $R_\sqsubseteq$ we have $aR_\sqsubseteq k R_< c$, and therefore $(a, a') <^* (c, c')$. All other cases are handled analogously. Proof of right monotonicity is similar. We conclude, therefore, that the relations respect left and right monotonicity.

Thus, we conclude that the product structure $\mathcal{F} \times \mathcal{Q}$ is a minimal interval structure. ∎

Now the theorem follows immediately. Suppose $\mathcal{M}, x \models \phi^+$. For any $(x, x') \in f^{-1}(x)$ we have $\mathcal{M}^*, (x, x') \models \phi^+$, and since f is surjective, there is at least one such (x, x'). Thus $\mathcal{M}^*$ is a model of ϕ^+, and by claim 3 it has the structure we want. ∎

Corollary 7.4
ϕ is satisfiable in $\mathcal{M}^*$.

Corollary 7.5
The logic of minimal interval structures is decidable.

Proof. In order to check whether an input formula ϕ is satisfiable in a minimal interval structure, it suffices to enumerate all pre-interval structures of at most size 2^{bn}. If we find a structure in which ϕ^+ is satisfied we output "ϕ satisfiable"; if not, then we output "ϕ unsatisfiable". The former condition is correct by Theorem 7.3 and corollary 7.4; and the latter is correct by Lemma 6.4. ∎

8 Complexity of the satisfiability problem for minimal interval structures

We give a construction of a deterministic exponential algorithm for the satisfiability of the logic of minimal interval structures. Again, by appealing to the bulldozing technique, we need only consider pre-interval structures. The complexity upper bound will be obtained by giving a modified version of a deterministic exponential algorithm first presented in [21]. The corresponding lower bound will be a straightforward reduction to the satisfiability problem over transitive frames for the Priorean tense language expanded with just one nominal (and no @ operator).

Theorem 8.1
The satisfiability problem for the logic of minimal interval structures is in EXPTIME.

Proof. By Lemma 6.2 and Lemma 6.3 (Truth Lemma) we established that ϕ^+ is satisfiable if and only if there is a ϕ^+-saturated set $\mathfrak{N}$. Let $\mathfrak{N}_0 \subseteq \mathcal{P}(\Phi_{\phi^+})$ consist of all S which are nice sets. We will construct a sequence of sets $\mathfrak{N}_0 \supset \mathfrak{N}_1 \supset \ldots$ such that: if ϕ^+ is satisfiable in a model $\mathcal{M}$, and $\mathfrak{N}$ is a ϕ^+-saturated set such that $\mathfrak{N} \subseteq \mathfrak{N}_0$, then $\mathfrak{N} \subseteq \mathfrak{N}_k$ for all k. Call a set $S \in \mathfrak{N}_k$ *defective* with respect to $\mathfrak{N}_k$ if the following occurs:

1. $\exists \Box\psi \in \Phi_{\phi^+}$ such that $\Box\psi \notin S$ (for some $\Box \in \{[F], [P], [D], [U]\}$) but there is *no* $S' \in \mathfrak{N}_k$ with $\psi \notin S'$ and $SR^\dagger_\theta S'$ (where $\theta \in \{<, >, \sqsubseteq, \sqsupseteq\}$ corresponding to $\Box$).

Now we can present our deterministic exponential algorithm for the satisfiability of ϕ^+. Let $i_1, i_2, \ldots, i_m$ denote the nominals in ϕ^+, then $m \leq n$. Let $f_1, f_2, \ldots, f_j$ be the set of all f

such that f is a mapping from $\{i_1, \ldots, i_m\}$ to $\mathfrak{N}_0$ such that $i_p \in f(i_p)$ for all $p = 1, \ldots, m$. So $j = (2^{bn})^m \leq 2^{bn^2}$. Then

procedure Start with $\mathfrak{N}_0$ and repeat for $l = 1, 2, \ldots, j$

1. **Outer loop**: Set $S^{i_p} = f_l(i_p)$ for $p = 1, 2, \ldots, m$ and
   let $\mathfrak{N}_1$ be the result of deleting from $\mathfrak{N}_0$ any sets S such that
   (a) $i_p \in S$ but $S \neq S^{i_p}$,
   (b) $\psi \in S^{i_p}$ but $@_{i_p}\psi \notin S$, and
   (c) $@_{i_p}\psi \in S$ but $\psi \notin S^{i_p}$
2. **Inner loop**: repeat for $k = 1$ up to 2^{bn}
   Let $\mathfrak{N}' = \mathfrak{N}_k$
   delete from $\mathfrak{N}'$ any set $S \in \mathfrak{N}'$ such that S is defective with respect to $\mathfrak{N}_k$
3. **if** $\nexists S \in \mathfrak{N}'$ with $\phi \in S$ *or* $S^{i_p} \notin \mathfrak{N}'$, for some nominals i_p, **then**
   (a) **if** $l = j$ then **fail**;
   (b) **otherwise** increment l and return to 1;
4. **if** $\mathfrak{N}_k = \mathfrak{N}'$ then **succeed**;
5. **otherwise** Increment k and return to 2.

That the algorithm is correct is shown as follows. Assume ϕ^+ is satisfiable. By Lemma 6.2, we know there is a ϕ^+-saturated set $\mathfrak{N} \subseteq \mathfrak{N}_0$ for which there is an f_l such that $f_l(i_p) \in \mathfrak{N}$ for all p. For such an f_l, any $S \in \mathfrak{N}$ will *not* be deleted from $\mathfrak{N}_0$, and therefore $\mathfrak{N} \subseteq \mathfrak{N}_1$. We will show by induction that $\mathfrak{N} \subseteq \mathfrak{N}_k$, for all k. Suppose, $\mathfrak{N} \subseteq \mathfrak{N}_{k-1}$, we will show that $\mathfrak{N} \subseteq \mathfrak{N}_k$. Since $\mathfrak{N} \subseteq \mathfrak{N}_{k-1}$ and $\mathfrak{N}$ is ϕ^+-saturated, at the k-th iteration, no $S \in \mathfrak{N}$ is defective with respect to $\mathfrak{N}_{k-1}$, and therefore no $S \in \mathfrak{N}$ will be deleted. Thus $\mathfrak{N} \subseteq \mathfrak{N}_k$. This means that clause 3 of the algorithm will not hold. And since, after each iteration $\mathfrak{N}_{k+1}$ is strictly included in $\mathfrak{N}_k$, the algorithm must terminate with $\mathfrak{N} \subseteq \mathfrak{N}_k$. Therefore, eventually clause 4 holds and the algorithm succeeds. Now, assume the algorithm terminates with success for some f_l and some $\mathfrak{N}_k$, in this case we want to show that ϕ^+ is satisfiable. We do this by checking that the conditions for saturation on $\mathfrak{N}_k$, given in section 6.1, are satisfied. Clearly, condition 1 is satisfied, since the algorithm terminated with success there must be a $S \in \mathfrak{N}_k$ with $\phi^+ \in S$. Similarly, condition 2 is satisfied, since the algorithm only succeeds if it does not delete any $S \in \mathfrak{N}_k$, and this only happens if no S is defective. Also, the satisfaction of condition 3 is given firstly, by clause 1(a) of the algorithm which ensures that each S^{i_p} is unique, and then by clause 3, which ensures that if any S^{i_p} is deleted, then the algorithm either fails or moves onto f_{l+1}. So, it remains to check that the 4th saturation condition is satisfied. Suppose $\psi \in S^{i_p}$, then by clause 1(b), we delete from $\mathfrak{N}_0$ any S such that $@_i\psi \notin S$. Thus, for any $S \in \mathfrak{N}_k$, we have $@_i\psi \in S$. If $@_i\psi \in S$ for all $S \in \mathfrak{N}_k$, then it follows immediately that $@_i\psi \in S$ for some $S \in \mathfrak{N}_k$. Finally, suppose $@_i\psi \in S$ for some $S \in \mathfrak{N}_k$, we want to show that $\psi \in S^{i_p}$. Suppose, for proof by contradiction, that $\psi \notin S^{i_p}$. Then by clause 1(c), we delete from $\mathfrak{N}_0$ any S such that $@_i\psi \in S$. Thus, for any $S \in \mathfrak{N}_k$ we have $@_i\psi \notin S$, which is a contradiction. We therefore conclude that $\mathfrak{N}_k$ is a ϕ^+-saturated set, and so by Lemma 6.3 (Truth Lemma) we have that ϕ^+ is satisfiable.

The set $\{f_1, f_2, \ldots, f_j\}$ is of exponential size, and therefore the outer loop will stop after exponentially many circles. Determining which sets to delete from $\mathfrak{N}_0$ takes polynomial time in the length of $\mathfrak{N}_0$. Therefore, for every member of $\mathfrak{N}_0$, the algorithm takes at most deterministic exponential time. For the inner loop, since $\mathfrak{N}_0$ is of exponential size, and after each iteration $\mathfrak{N}_{k+1}$ is strictly included in $\mathfrak{N}_k$, the algorithm terminates after at most exponentially many circles. Again determining which sets in $\mathfrak{N}_k$ are defective takes polynomial time in the length of the representation of $\mathfrak{N}_k$. Thus, for every member of $\mathfrak{N}_k$, the algorithm takes at most deterministic exponential time. Since both $\mathfrak{N}_0$ and $\{f_1, f_2, \ldots, f_j\}$ are of exponential size, we can determine if ϕ^+ is satisfiable in EXPTIME. ∎

THEOREM 8.2
The satisfiability problem for the logic of minimal interval structures in the interval hybrid language with just one nominal (and no @ operator) is EXPTIME-hard.

PROOF. By reduction to the satisfiability problem over transitive frames for the Priorean tense language expanded with just one nominal (and no @ operator). The Priorean tense language is built using only the operators $\langle F \rangle$ and $\langle P \rangle$. The EXPTIME-completeness of this problem is proved in [3]. The translation function $(\cdot)^t$ from Priorean tense formulas to formulas in the interval hybrid language is simply the identity function. Obviously it is a linear reduction.

CLAIM: For any Priorean tense formula ϕ, ϕ is satisfiable in a hybrid model based on a transitive frame iff ϕ^t is satisfiable in a hybrid model based on a minimal interval structure.

PROOF OF CLAIM.

[⇒]. Suppose $\mathcal{M} \models \phi$, where $\mathcal{M} = (W, R_<, R_>, V)$. We now define a model $\mathcal{M}^t = (W^t, R^t_<, R^t_>, R^t_\sqsubseteq, R^t_\sqsupseteq, V^t)$ based on a minimal interval structure as follows:

1. $W^t = W$
2. $R^t_< = R_<$
3. $R^t_> = R_>$
4. $R^t_\sqsubseteq = \{(w_0, w_0) : w_0 \in W\} = R^t_\sqsupseteq$
5. $V^t = V$

In particular, for nominal i, $V^t(i) = V(i)$.

It follows by induction that for all $w \in W$, for all Priorean tense formulas ψ, $\mathcal{M}, w \models \psi$ iff $\mathcal{M}^t, w \models \psi$. All cases, including the modalities, are straightforward. It remains to check that each of the relations satisfy the appropriate conditions. Since, both $R_<$ and $R_>$ are transitive, the transitivity of $R^t_<$ and $R^t_>$ are immediate. The reflexivity and transitive of $R^t_\sqsubseteq$ and $R^t_\sqsupseteq$ also follow immediately from our construction. Left and right monotonicity is also immediate, since $R^t_\sqsupseteq$ is simply the identity relation.

[⇐]. Suppose $\mathcal{M}^t \models \phi^t$, where $\mathcal{M}^t = (W^t, R^t_<, R^t_>, R^t_\sqsubseteq, R^t_\sqsupseteq, V^t)$ is a model based on a minimal interval structure. We define a hybrid transitive model $\mathcal{M} = (W, R_<, R_>, V)$ as follows:

1. $W = W^t$
2. $R_< = R^t_<$
3. $R^t_> = R_>$
4. $V = V^t$

It follows by induction that for all $w \in W^t$, for all interval hybrid formulas ψ, $\mathcal{M}^t, w \models \psi$ iff $\mathcal{M}, w \models \psi$. Again, all the cases, including the modalities are straightforward.

The theorem follows directly from the claim. ∎

COROLLARY 8.3
The satisfiability problem for the logic of minimal interval structures is EXPTIME-complete.

PROOF. Follows immediately from Theorem 8.1 and Theorem 8.2. ∎

REMARK 8.4
The satisfiability problem for the logic of minimal interval structures in the interval hybrid temporal language in which the @ operator is replaced by the somewhere modality $\mathbf{E}$ is solvable in EXPTIME. The interpretation of $\mathbf{E}$ is *fixed*: in any model $\mathcal{M} = (W, R, V)$, $\mathbf{E}$ must be interpreted using the relation $W \times W$. That is:

$\mathcal{M}, w \models \mathbf{E}\phi$ iff there is a $u \in W$ such that $\mathcal{M}, u \models \phi$.

Thus $\mathbf{E}$ scans the entire model for a state that satisfies ϕ. The formula $@_i\phi$ is then equivalent to $\mathbf{E}(i \wedge \phi)$. That the satisfiability problem for the interval hybrid temporal language with $\mathbf{E}$ (and without @) is in EXPTIME can be shown by modifying the deterministic exponential algorithm of Theorem 8.1. The existential aspect of $\mathbf{E}$ is treated as a defect in the inner loop of the algorithm (stage 2), whilst the global aspect is checked at the end of the inner loop (stage 3).

9 Concluding remarks

In [13] it was shown that the satisfiability problem for interval structures depended critically on our underlying assumptions about time. For many interesting classes of structures, the satisfiability problem is undecidable. In particular, they showed that, if we take our underlying temporal structure to be the rationals, then the satisfiability problem is r.e-complete; if it is the reals, then it's Π^1_1-hard; and if it's the natural numbers, then satisfiability is Π^1_1-complete. Following on from the work in [17], we introduce an interval hybrid temporal logic for talking about interval temporal structures. We study the logic in its full generality and identify two simple classes of interval temporal structures structures. Unlike the case for the interval temporal logic in [17], we show that the interval hybrid temporal logic is able to distinguish between the logic of minimal interval structures and the logic of van Benthem minimal interval structures. We go on to prove that, for the class of minimal interval structures, the satisfiability problem is EXPTIME-complete in the interval hybrid temporal logic (without $\downarrow$). We obtain the necessary decidability result by developing a novel bulldozing technique that not only handles interacting relations (and therefore can generalize straightforwardly to modal languages), but also works in the presence of nominals. A number of interesting questions arise from our work which would be worthwhile pursuing:

1. The most interesting open question raised by this paper concerns the logic of interval structures with the conjectivity condition. While we have a sound and complete tableau system for this logic, it is unclear whether this logic is decidable and (if it is) what its complexity is. It is unclear whether the bulldozing method used in this paper can be adapted to prove decidability for this logic.
2. What, if any, further assumptions are needed to obtain concrete interval temporal structures? And what further assumptions about time (such as convexity, linearity, density etc.) can we make whose addition/inclusion makes the satisfiability problem (un)decidable?
3. Which, if any, other modalities can we incorporate into our language without losing decidability?

We leave the investigation of these questions for further work.

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