Laplace Transform Inversion and Passage-Time Distributions in Markov Processes

Peter G. Harrison

Journal of Applied Probability, Vol. 27, No. 1. (Mar., 1990), pp. 74-87.

Stable URL:
http://links.jstor.org/sici?sici=0021-9002%28199003%2927%3A1%3C74%3ALTIAPD%3E2.0.CO%3B2-P

Journal of Applied Probability is currently published by Applied Probability Trust.

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at http://www.jstor.org/about/terms.html. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at http://www.jstor.org/journals/apt.html.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

The JSTOR Archive is a trusted digital repository providing for long-term preservation and access to leading academic journals and scholarly literature from around the world. The Archive is supported by libraries, scholarly societies, publishers, and foundations. It is an initiative of JSTOR, a not-for-profit organization with a mission to help the scholarly community take advantage of advances in technology. For more information regarding JSTOR, please contact support@jstor.org.

http://www.jstor.org
Tue Mar 25 19:46:53 2008

J. Appl. Prob. **27**, 74–87 (1990)
Printed in Israel
© *Applied Probability Trust* 1990

LAPLACE TRANSFORM INVERSION AND PASSAGE-TIME DISTRIBUTIONS IN MARKOV PROCESSES

PETER G. HARRISON,* *Imperial College, London*

Abstract

Products of the Laplace transforms of exponential distributions with different parameters are inverted to give a mixture of Erlang densities, i.e. an expression for the convolution of exponentials. The formula for these inversions is expressed both as an explicit sum and in terms of a recurrence relation which is better suited to numerical computation. The recurrence for the inversion of certain weighted sums of these transforms is then solved by converting it into a linear first-order partial differential equation. The result may be used to find the density function of passage times between states in a Markov process and it is applied to derive an expression for cycle time distribution in tree-structured Markovian queueing networks.

ANALYTICAL INVERSION OF LAPLACE TRANSFORMS; TIME DELAY DENSITY; CYCLE TIME; RESPONSE TIME; QUEUEING NETWORKS

1. Introduction

For many problems in stochastic modelling that involve the distribution or density, if it exists, of a random variable, solutions are expressed in the form of the Laplace–Stieltjes or Laplace transform; see for example queueing network and Brownian motion applications such as Schassberger and Daduna (1983), Kelly and Pollett (1983), Harrison (1986), Harrison (1985). Whilst being of value in itself, for example as a source of the moments of the required distribution, the Laplace transform is inadequate for problems which require estimates for related probabilities, for example in reliability modelling. Apart from simple cases where inversion can be performed by inspection, e.g. for a single Erlang distribution, the most common approach to deriving distributions in the time domain from transforms is by numerical inversion, see for example Davies and Martin (1979), Abate and Whitt (1988). Unfortunately, however, in view of the 'smoothing' effect of the Laplace transform operation, arising from the smoothness of the infinitely differentiable exponential function, numerical methods are not easy to implement accurately, especially where the tail of a distribution is concerned — often the most important region.

The present paper provides a technique for inverting analytically Laplace transforms formed from products of the form

Received 20 October 1988; revision received 6 March 1989.
* Postal address: Department of Computing, Imperial College, 180 Queen's Gate, London SW7 2BZ, UK.

$$\left[\frac{\mu_1}{s+\mu_1}\right]^{n_1}\cdots\left[\frac{\mu_m}{s+\mu_m}\right]^{n_m},$$

i.e. transforms of finite convolutions of negative exponential densities. The result is expressed as a mixture of Erlang densities and since the Laplace transform operator is distributive over addition, this means that such expressions can be obtained for Coxian distributions. However, perhaps the most important application is computing the distribution of passage times betwen states in a Markov process. Since every transition occurs at a Markov time, the distribution corresponding to any given sequence of states is a convolution of exponentials. Hence the passage-time distribution is a weighted sum of the Laplace transform inversions corresponding to the sequences of states entered between the given initial and final states, the weights being the selection probabilities of these sequences.

In the next section the inversion formula is given, first in the form of an explicit sum of Erlang densities and then as a recurrence relation which is better suited to numerical computation. The recurrence relation for a particular type of weighted sum of products over a domain corresponding to the state space of a closed queueing network is then solved by finding an equivalent linear first-order partial differential equation (p.d.e.). In Section 3 this result is applied to derive an expression for cycle time distribution in tree-structured Markovian queueing networks and the paper concludes in Section 4 with a summary and suggestions for future work.

2. The inversion formula for a convolution of exponentials

The inverse of the Laplace transform of a finite convolution of exponential distributions is a mixture of Erlang distributions, the density for which is given by the following theorem.

Theorem 1. The probability density function which has Laplace transform

$$L(\boldsymbol{n},s)=\prod_{i=1}^{M}\left[\frac{\mu_i}{s+\mu_i}\right]^{n_i}$$

where $\boldsymbol{n}=(n_1,\cdots,n_M)$, $n_i\geqq 1$ and the μ_i's are distinct ($1\leqq i\leqq M$) is

$$f(\boldsymbol{n},t)=\left[\prod_{i=1}^{M}\mu_i^{n_i}\right]\sum_{j=1}^{M}D_j(\boldsymbol{n},t)$$

where

$$D_j(\boldsymbol{n},t)=\frac{(-1)^{n_j-1}\exp(-\mu_j t)}{\prod_{\substack{i=1\\ i\neq j}}^{M}(n_i-1)!}\sum_{\substack{\Sigma_{i=1}^{M}k_i=n_j-1\\ k_i\geqq 0}}\left\{\frac{(-t)^{k_j}}{k_j!}\prod_{\substack{i=1\\ i\neq j}}^{M}\frac{(n_i+k_i-1)!}{(\mu_i-\mu_j)^{n_i+k_i}k_i!}\right\}.$$

Before the proof, the following example should help to clarify the notation. Let the density $f=E_{\mu_1}(3)*E_{\mu_2}(1)$ where $E_\lambda(k)$ denotes the Erlang-k density with parameter λ and $*$ denotes convolution. Then in the theorem we have $\boldsymbol{n}=(3,1)$ and $M=2$ so that $f(t)=\mu_1^3\mu_2(D_1+D_2)$ where

$$D_1 = \frac{\exp(-\mu_1 t)}{2}\left[\frac{t^2}{2(\mu_2-\mu_1)} - \frac{t}{(\mu_2-\mu_1)^2} + \frac{1}{(\mu_2-\mu_1)^3}\right]$$

since for $j = 1$, $n_j = 3$ and so $\boldsymbol{k}$ is summed over the domain $\{(2,0), (1,1), (0,2)\}$, and

$$D_2 = \frac{\exp(-\mu_2 t)}{(\mu_1-\mu_2)^3}$$

since for $j = 2$, $n_j = 1$ and $\boldsymbol{k}$ is summed over the domain $\{(0,0)\}$.

Proof of Theorem 1. The Laplace transform, $L(\boldsymbol{n}, s)$, is inverted by evaluation of the Bromwich contour integral, to give

$$f(\boldsymbol{n}, t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} L(\boldsymbol{n}, s)\,ds$$

where $\gamma > \gamma_0 = \mathrm{Re}(s_1)$ and s_1 is the singularity of $L(\boldsymbol{n}, s)$ with the largest real part. Since all singularities in $L(\boldsymbol{n}, s)$ are isolated poles,

$$f(\boldsymbol{n}, t) = \sum \text{residues of } \{e^{st} L(\boldsymbol{n}, s)\} \text{ at poles}$$

where the residue of a function $G(s)$ at s_0 is the coefficient of $(s - s_0)^{-1}$ in the Laurent expansion of $G(s)$ about s_0. Thus

$$f(\boldsymbol{n}, t) = \sum_{j=1}^{M} \text{residues of } \prod_{i=1}^{M}\left[\frac{\mu_i}{s+\mu_i}\right]^{n_i} e^{st} \text{ at } s = -\mu_j.$$

Now, if we define

$$\Delta_j(\boldsymbol{n}, t) = \frac{1}{(n_j - 1)!}\lim_{s\to -\mu_j}\left\{\frac{d^{n_j - 1}}{ds^{n_j - 1}}\left[\frac{e^{st}}{\Pi_{i\neq j}(s+\mu_i)^{n_i}}\right]\right\}$$

then $f(\boldsymbol{n}, t) = [\Pi_{i=1}^{M} \mu_i^{n_i}]\Sigma_{j=1}^{M} \Delta_j(\boldsymbol{n}, t)$ since the pole at $s = -\mu_j$ has order n_j by the hypothesis that the μ_j's are distinct $(1 \leqq j \leqq M)$.

Using the generalisation of Leibnitz's rule for differentiation of a multiple product,

$$\Delta_j(\boldsymbol{n}, t) = \lim_{s\to -\mu_j} \sum_{\substack{\Sigma_{i=1}^{M} k_i = n_j - 1 \\ k_i \geqq 0}} \frac{1}{\prod_{i=1}^{M} k_i!} e^{st} t^{k_j} \prod_{\substack{i=1 \\ i\neq j}}^{M} \frac{(-1)^{k_i} n_i (n_i+1)\cdots(n_i+k_i-1)}{(s+\mu_i)^{n_i+k_i}}$$

$$= D_j(\boldsymbol{n}, t)$$

as required since $\Pi_{i\neq j}(-1)^{k_i} = (-1)^{n_j - 1 - k_j}$.

Corollary. For $n_j \geqq 1$ $(1 \leqq j \leqq M)$, let $\boldsymbol{n}^j = (n_1, \cdots, n_j - 1, \cdots, n_M)$, $\boldsymbol{n}^{jk} = (n_1, \cdots, n_j - 1, \cdots, n_k + 1, \cdots, n_M)$. Then

$$f(\boldsymbol{n}, t) = \left[\prod_{i=1}^{M} \mu_i^{n_i}\right] \sum_{j=1}^{M} D_j(\boldsymbol{n}, t)$$

where $D_j(\boldsymbol{n}, t)$ is given by the recurrence formula:

$$(n_j - 1)D_j(\mathbf{n}, t) = tD_j(\mathbf{n}^j, t) - \sum_{\substack{k=1 \\ k \neq j}}^{M} n_k D_j(\mathbf{n}^{jk}, t)$$

with boundary condition

$$D_j(\mathbf{n}, t) = \frac{\exp(-\mu_j t)}{\prod_{\substack{i=1 \\ i \neq j}}^{M} (\mu_i - \mu_j)^{n_i}} \qquad (n_j = 1, n_i \geqq 1, 1 \leqq i \neq j \leqq M).$$

Proof. Differentiating once with respect to s and rearranging, we obtain the following expression for $D_j(\mathbf{n}, t)$:

$$D_j(\mathbf{n}, t) = \frac{t}{n_j - 1} \frac{1}{(n_j - 2)!} \lim_{s \to -\mu_j} \left\{ \frac{d^{n_j - 2}}{ds^{n_j - 2}} \left[\frac{e^{st}}{\Pi_{i \neq j} (s + \mu_i)^{n_i}} \right] \right\}$$

$$- \sum_{k \neq j} \frac{n_k}{n_j - 1} \frac{1}{(n_j - 2)!} \lim_{s \to -\mu_j} \left\{ \frac{d^{n_j - 2}}{ds^{n_j - 2}} \left[\frac{e^{st}}{(s + \mu_k) \Pi_{i \neq j} (s + \mu_i)^{n_i}} \right] \right\}.$$

Thus $(n_j - 1)D_j(\mathbf{n}, t) = tD_j(\mathbf{n}^j, t) - \Sigma_{k \neq j} n_k D_j(\mathbf{n}^{jk}, t)$.

Finally, for $n_j = 1$,

$$D_j(\mathbf{n}, t) = \lim_{s \to -\mu_j} \left\{ e^{st} \prod_{i \neq j} (s + \mu_i)^{-n_i} \right\} = \exp(-\mu_j t) \prod_{i \neq j} (\mu_i - \mu_j)^{-n_i}.$$

In practice, it is not generally a single product of the form

$$\left[\frac{\mu_1}{s + \mu_1}\right]^{n_1} \cdots \left[\frac{\mu_m}{s + \mu_m}\right]^{n_m}$$

that needs to be inverted but a *mixture*, that is a weighted sum of the products. Such a mixture might derive from the possible sequences of states entered by a process between predefined sets of initial and final states; the weights are the probabilities that each sequence is selected, a product of state transition probabilities. Theorems 2 and 3 below invert geometrically weighted sums over the summation domains

$$S_{MN} = \left\{ \mathbf{n} \;\middle|\; \sum_{i=1}^{M} n_i = N; n_i \geqq 1, 1 \leqq i \leqq M \right\} \quad \text{and} \quad T_M = \{\mathbf{n} \mid n_i \geqq 1, 1 \leqq i \leqq M\}.$$

(In the rest of this section we will abbreviate S_{MN}, T_M by S_N, T respectively.) The result for the former, finite case is then applied in the next section to find the distribution of passage times through tree-structured networks. To prove the theorems we introduce generating functions which are determined in corresponding lemmas. We begin with the finite case, a weighted sum over S_{MN}.

Lemma 1. Given real numbers $a_1, \cdots, a_M$ let the M-parameter generating function H_{jm} be defined by

$$H_{jm}(\mathbf{z}) = \sum_{\mathbf{n} \in S_{M+m}} D_j(\mathbf{n}, t) \prod_{i=1}^{M} (a_i z_i)^{n_i - 1} \qquad (1 \leqq j \leqq M, m \geqq 0)$$

where $z = (z_1, \cdots, z_M)$. Then

$$H_{jm}(z) = \frac{\exp(-\mu_j t)}{\prod_{1 \leq i \neq j \leq M} (\mu_i - \mu_j)} \left\{ \sum_{i=0}^{m} \frac{(a_j z_j t)^{m-i}}{(m-i)!} \sum_{\substack{n \in S_{M+i} \\ n_j = 1}} \prod_{\substack{k=1 \\ k \neq j}}^{M} \left[\frac{(a_k z_k - a_j z_j)}{\mu_k - \mu_j} \right]^{n_k - 1} \right\}.$$

(Throughout the rest of this lemma and its corollaries we drop the suffix j from H.)

Proof. By multiplying both sides of the recurrence relation given in the Corollary to Theorem 1 by $\prod_{i=1}^{M} (a_i z_i)^{n_i - 1}$ and rearranging slightly, we obtain, for $n_j > 1$, $1 \leq j \leq M$:

$$(n_j - 1) D_j(n, t) \prod_{i=1}^{M} (a_i z_i)^{n_i - 1}$$

$$= t D_j(n^j, t) a_j z_j \prod_{i=1}^{M} (a_i z_i)^{n_i^j - 1} - \sum_{1 \leq j \neq k \leq M} \frac{n_k}{a_k z_k} a_j z_j D_j(n^{jk}, t) \prod_{i=1}^{M} (a_i z_i)^{n_i^{jk} - 1}.$$

We now sum both sides over $n \in S_N$ for $n_j \geq 2$, changing the summation variables on the right-hand side. Notice that as n ranges over S_N with $n_j \geq 2$ $(1 \leq j \leq M)$ n^j ranges over the whole of S_{N-1} and $s = n^{jk}$ $(1 \leq j \neq k \leq M)$ may be assumed to range over the whole of S_N if terms corresponding to $s_k = 1$ give zero contribution, which they do in view of the factor $n_k = s_k - 1$. We therefore obtain, for $N > M$:

$$\sum_{n \in S_N} \frac{n_j - 1}{a_j z_j} d_{jN}(n) = \sum_{m \in S_{N-1}} t d_{j,N-1}(m) - \sum_{\substack{s \in S_N \\ 1 \leq k \neq j \leq M}} \frac{s_k - 1}{a_k z_k} d_{jN}(s)$$

where

$$d_{jN}(n) = D_j(n, t) \prod_{i=1}^{M} (a_i z_i)^{n_i - 1} \quad \text{for } n \in S_N.$$

(On the left-hand side, summands with $n_j = 1$ give zero contribution.) Rearranging now gives:

$$\sum_{n \in S_N} \left\{ \sum_{k=1}^{M} \frac{n_k - 1}{a_k z_k} \right\} D_j(n, t) \prod_{i=1}^{M} (a_i z_i)^{n_i - 1} = t H_{N-M-1}(z).$$

Letting $n = N - M$, this is equivalent to the partial differential difference equation

$$\sum_{k=1}^{M} \frac{1}{a_k} \frac{\partial H_n(z)}{\partial z_k} = t H_{n-1}(z) \qquad (n > 0)$$

with

$$H_0(z) = D_j((1, 1, \cdots, 1), t) = \frac{\exp(-\mu_j t)}{\prod_{1 \leq i \neq j \leq M} (\mu_i - \mu_j)}$$

which is constant in z.

This is a first-order linear p.d.e. that can be solved using Laplace's method which first finds M independent solutions to the equations:

$$a_1 dz_1 = \cdots = a_M dz_M = \frac{dH_n}{tH_{n-1}}.$$

For the given value of j, clearly $M-1$ independent solutions are $a_i z_i - a_j z_j = 0$ $(1 \leqq i \neq j \leqq M)$ and if $H_n(\mathbf{z}) = u(\mathbf{z})$ is another, the general solution of the p.d.e. is:

$$H_n(\mathbf{z}) - u(\mathbf{z}) = \psi_n(a_1 z_1 - a_j z_j, \cdots, a_{j-1} z_{j-1} - a_j z_j, a_{j+1} z_{j+1} - a_j z_j, \cdots, a_M z_M - a_j z_j)$$

which we abbreviate to $\psi_n(a_i z_i - a_j z_j \mid 1 \leqq i \neq j \leqq M)$ for an arbitrary $(M-1)$ary function ψ_n. Consider first the case $n = 1$. Here $u = \int tH_0 a_j dz_j$ (choosing z_j arbitrarily for the integration) and since H_0 is a constant, we have:

$$H_1(\mathbf{z}) = tH_0 a_j z_j + \psi_1(a_i z_i - a_j z_j \mid i \neq j)$$

where $\psi_1(a_i z_i - a_j z_j \mid i \neq j)$ is determined by the boundary condition at $z_j = 0$, as below.

By repeating this process we can compute $H_m(\mathbf{z})$ from $H_{m-1}(\mathbf{z})$ successively for $m = 2, 3, \cdots$. In the integrations of $H_{m-1}(\mathbf{z})$ we note that

$$\int \psi_n(a_i z_i - a_j z_j \mid i \neq j) z_j^k dz_j = \psi_n(a_i z_i - a_j z_j \mid i \neq j) \frac{z_j^{k+1}}{k+1}$$

since $a_i dz_i = a_j dz_j$. ($a_i dz_i = a_j dz_j$ for $i \neq j$ implies that the differential $d\psi_n(a_i z_i - a_j z_j \mid i \neq j) = 0$, i.e. $\psi_n(a_i z_i - a_j z_j \mid i \neq j)$ is a constant; alternatively integrate the integrand by parts using $d\psi_n = 0$.)

In this way we obtain:

$$H_m(\mathbf{z}) = \frac{t^m (a_j z_j)^m}{m!} H_M + \sum_{k=0}^{m-1} \frac{t^k (a_j z_j)^k}{k!} \psi_{m-k}(a_i z_i - a_j z_j \mid 1 \leqq i \neq j \leqq M)$$

where the boundary condition at $z_j = 0$ yields the identity:

$$\begin{aligned} \psi_m(a_i z_i \mid 1 \leqq i \neq j \leqq M) &= H_m(z_1, \cdots, z_{j-1}, 0, z_{j+1}, \cdots, z_M) \\ &= \sum_{\substack{\mathbf{n} \in S_{M+m} \\ n_j = 1}} D_j(\mathbf{n}, t) \prod_{\substack{k=1 \\ k \neq j}}^{M} (a_k z_k)^{n_k - 1}. \end{aligned}$$

Since

$$D_j(\mathbf{n}, t) = \frac{\exp(-\mu_j t)}{\prod_{1 \leqq i \neq j \leqq M} (\mu_i - \mu_j)^{n_i}}$$

when $n_j = 1$, this simplifies to:

$$\psi_m(a_i z_i \mid 1 \leqq i \neq j \leqq M) = \frac{\exp(-\mu_j t)}{\prod_{1 \leqq i \neq j \leqq M} (\mu_i - \mu_j)} \left\{ \sum_{\substack{\mathbf{n} \in S_{M+m} \\ n_j = 1}} \prod_{\substack{k=1 \\ k \neq j}}^{M} \left[\frac{a_k z_k}{\mu_k - \mu_j} \right]^{n_k - 1} \right\}.$$

This fully defines the function ψ and we obtain:

$$H_m(\mathbf{z}) = \frac{\exp(-\mu_j t)}{\prod\limits_{1 \leqq i \neq j \leqq M} (\mu_i - \mu_j)} \left\{ \sum_{i=0}^{m} \frac{t^{m-i}(a_j z_j)^{m-i}}{(m-i)!} \sum_{\substack{\mathbf{n} \in S_{M+i} \\ n_j = 1}} \prod_{\substack{k=1 \\ k \neq j}}^{M} \left[\frac{(a_k z_k - a_j z_j)}{\mu_k - \mu_j} \right]^{n_k - 1} \right\}$$

since when $i = 0$, S_{M+i} contains the single element $(1, \cdots, 1)$ so that the sum of the products over S_{M+i} is 1 as required.

Corollary 1.

$$H_m(\mathbf{z}) = \frac{\exp(-\mu_j t)}{\prod\limits_{1 \leqq i \neq j \leqq M} (\mu_i - \mu_j)} \sum_{i=0}^{m} \frac{(a_j z_j t)^{m-i}}{(m-i)!} G_j(M-1, i) \qquad (m \geqq 0)$$

where

$$G_j(h, k) = G_j(h-1, k) + y_h G_j(h, k-1) \qquad (h, k \geqq 1)$$

$$G_j(h, 0) = 1 \qquad (h \geqq 0), \quad G_j(0, k) = 0 \qquad (k > 0)$$

and

$$y_i = \begin{cases} \dfrac{(a_i z_i - a_j z_j)}{\mu_i - \mu_j} & (1 \leqq i < j) \\[2ex] \dfrac{(a_{i+1} z_{i+1} - a_j z_j)}{\mu_{i+1} - \mu_j} & (j \leqq i \leqq M-1). \end{cases}$$

Proof. It is sufficient to show that

$$G_j(h, k) = \sum_{\substack{\mathbf{n} \in S_{h+k+1} \\ n_j = 1}} \prod_{i=1}^{h} y_i^{n_i - 1} \qquad (h, k \geqq 0).$$

If we define the set of vectors

$$V_{mn} = \left\{ \mathbf{n} \mid n_i \geqq 0, 1 \leqq i \leqq m; \sum_{i=1}^{m} n_i = n \right\},$$

this is the same as showing that

$$G_j(h, k) = \sum_{\mathbf{n} \in V_{hk}} \prod_{i=1}^{h} y_i^{n_i}.$$

Following Buzen (1973), we can partition the domain of summation to get:

$$G_j(h, k) = \sum_{\substack{\mathbf{n} \in V_{hk} \\ n_h = 0}} \prod_{i=1}^{h} y_i^{n_i} + \sum_{\substack{\mathbf{n} \in V_{hk} \\ n_h > 0}} \prod_{i=1}^{h} y_i^{n_i}$$

$$= G_j(h-1, k) + y_h \sum_{\substack{\mathbf{n} \in V_{h,k-1} \\ n_h = 0}} \prod_{i=1}^{h} y_i^{n_i}$$

$$= G_j(h-1, k) + y_h G_j(h, k-1).$$

The boundary conditions are immediate, corresponding to an empty sum and product.

Corollary 2. If the ratios $(a_i z_i - a_j z_j)/(\mu_i - \mu_j)$ are distinct $(1 \leqq i \neq j \leqq M)$, then

$$H_m(z) = \frac{\exp(-\mu_j t)}{\prod\limits_{1 \le i \ne j \le M} (\mu_i - \mu_j)} \sum_{i=0}^{m} \frac{(a_j z_j t)^{m-i}}{(m-i)!} \sum_{\substack{u=1 \\ u \ne j}}^{M} \frac{(a_u z_u - a_j z_j)^{M+i-2}}{(\mu_u - \mu_j)^i}$$

$$\times \prod_{\substack{v=1 \\ v \ne u,j}}^{M} \frac{\mu_v - \mu_j}{a_u z_u \mu_v - a_v z_v \mu_u + a_j z_j(\mu_u - \mu_v) - \mu_j(a_u z_u - a_v z_v)}.$$

Proof. In the notation of the proof of Corollary 1, the y_i are distinct $(1 \le i \le M-1)$ and so by the result in Harrison (1984),

$$G_j(M-1, i) = \sum_{u=1}^{M-1} \frac{y_k^{M+i-2}}{\prod\limits_{\substack{v=1 \\ v \ne u}}^{M-1} (y_u - y_v)}.$$

The density function for the corresponding weighted sum of products of Laplace transforms is therefore given by the following result.

Theorem 2. Let $D(\boldsymbol{n}, t) = \Sigma_{j=1}^{M} D_j(\boldsymbol{n}, t)$. Then given positive real numbers $a_1, \cdots, a_M$,

$$\sum_{\boldsymbol{n} \in S_N} D(\boldsymbol{n}, t) \prod_{i=1}^{M} a_i^{n_i - 1} = \sum_{j=1}^{M} \frac{\exp(-\mu_j t)}{\prod\limits_{1 \le i \ne j \le M} (\mu_i - \mu_j)} \sum_{i=0}^{N-M} \frac{(a_j t)^{N-M-i}}{(N-M-i)!} G_j(M-1, i)$$

where for $h, k \ge 0$,

$$G_j(h, k) = G_j(h-1, k) + y_h G_j(h, k-1)$$

$$G_j(h, 0) = 1, \quad G_j(0, k) = 0 \qquad (k \ne 0)$$

and

$$y_i = \begin{cases} \dfrac{(a_i - a_j)}{\mu_i - \mu_j} & (1 \le i < j) \\[2ex] \dfrac{(a_{i+1} - a_j)}{\mu_{i+1} - \mu_j} & (j \le i \le M-1). \end{cases}$$

Proof. This follows immediately from Corollary 1 of Lemma 1 since

$$\sum_{\boldsymbol{n} \in S_N} D(\boldsymbol{n}, t) \prod_{i=1}^{M} a_i^{n_i - 1} = \sum_{j=1}^{M} H_{j, N-M}(1, \cdots, 1).$$

Corollary. If the ratios $(a_i - a_j)/(\mu_i - \mu_j)$ are distinct $(1 \le i \ne j \le M)$, then

$$\sum_{\mathbf{n}\in S_N} D(\mathbf{n},t)\prod_{i=1}^{M} a_i^{n_i-1}$$

$$=\frac{\exp(-\mu_j t)}{\prod_{1\leq i\neq j\leq M}(\mu_i-\mu_j)}\sum_{i=0}^{m}\frac{(a_i t)^{m-i}}{(m-i)!}\sum_{\substack{u=1\\u\neq j}}^{M}\frac{(a_u-a_j)^{M+i-2}}{(\mu_u-\mu_j)^i}$$

$$\times\prod_{\substack{v=1\\v\neq u,j}}^{M}\frac{\mu_v-\mu_j}{a_u\mu_v-a_v\mu_u+a_j(\mu_u-\mu_v)-\mu_j(a_u-a_v)}.$$

For small values of M, N the size of the set S_N is sufficiently small that the corollary does not provide a simpler computation. However, the size of S_N grows combinatorially whereas the complexity of the sum over u is only cubic in M; the complexity of the product is linear in M but there are $M(M-2)$ such products which can be precomputed.

Observation. Notice that in the special case that $a_j = 1$ for every j $(1 \leqq j \leqq M)$, the result simplifies to

$$\sum_{\mathbf{n}\in S_N} D(\mathbf{n},t)=\frac{t^{N-M}}{(N-M)!}\sum_{j=1}^{M}\frac{\exp(-\mu_j t)}{\prod_{1\leq i\neq j\leq M}(\mu_i-\mu_j)}.$$

This follows since we have, using similar reasoning to the proof of Lemma 1, the much simpler recurrence relation:

$$\sum_{\mathbf{n}\in S_N}(n_j-1)D_j(\mathbf{n},t)=\sum_{\mathbf{m}\in S_{N-1}} tD_j(\mathbf{m},t)-\sum_{\substack{\mathbf{s}\in S_N\\1\leq k\neq j\leq M}}(n_k-1)D_j(\mathbf{s},t)$$

$$(1\leqq j\leqq M, N>M).$$

But $\Sigma_{k\neq j}(n_k-1)=N-n_j-M+1$. Writing $A_j(K)$ for $\Sigma_{\mathbf{n}\in S_K} D_j(\mathbf{n},t)$, we therefore get $(N-M)\cdot A_j(N)=t\cdot A_j(N-1)$ for $N>M$, giving $A_j(N)=(t^{N-M}/(N-M)!)A_j(M)$ where $A_j(M)=D_j((1,1,\cdots,1),t)$ as before.

We now consider the rather simpler case of a similar weighted sum of products of Laplace transforms over the infinite set T. Again we proceed via a generating function given in the following.

Lemma 2. Given real numbers $a_1,\cdots,a_M$, let the M-parameter generating function K_j be defined by

$$K_j(\mathbf{z})=\sum_{\mathbf{n}\in T} D_j(\mathbf{n},t)\prod_{i=1}^{M}(a_i z_i)^{n_i-1}\qquad(1\leqq j\leqq M).$$

Then

$$K_j(\mathbf{z})=\frac{\exp((a_j z_j-\mu_j)t)}{\prod_{1\leq i\neq j\leq M}(\mu_i-\mu_j-a_i z_i+a_j z_j)}.$$

(Henceforth we will drop the suffix j from K.)

Proof. The initial part of the proof of Lemma 1 goes through unchanged and we arrive at the recurrence relation:

$$\sum_{n\in T}\left\{\sum_{k=1}^{M}\frac{n_k-1}{a_kz_k}\right\}D_j(n,t)\prod_{i=1}^{M}(a_iz_i)^{n_i-1}=tK(z).$$

This gives the p.d.e. $\Sigma_{k=1}^{M}(1/a_k)(\partial K/\partial z_k)=tK$, for the solution of which we require M independent solutions of the equations $a_1dz_1=\cdots=a_Mdz_M=dK/tK$ or equivalently of:

$$a_1dz_1=\cdots=a_Mdz_M=\frac{dL}{t}$$

where $L(z)=\log_e K(z)$. This has general solution $L=a_jz_jt+\psi(a_iz_i-a_jz_j\mid i\neq j)$ for an arbitrary function ψ, giving $K=\phi(a_iz_i-a_jz_j\mid i\neq j)\exp(a_jz_jt)$ for an arbitrary positive-valued function ϕ. The boundary condition at $z_j=0$ now gives:

$$\begin{aligned}\phi(a_iz_i\mid i\neq j)&=\frac{\exp(-\mu_jt)}{\prod_{1\le i\neq j\le M}(\mu_i-\mu_j)}\left\{\sum_{\substack{n_i\ge 1\,(i\neq j)\\ n_j=1}}\prod_{\substack{k=1\\k\neq j}}^{M}\left\{\frac{a_kz_k}{\mu_k-\mu_j}\right\}^{n_k-1}\right\}\\&=\frac{\exp(-\mu_jt)}{\prod_{1\le i\neq j\le M}(\mu_i-\mu_j)}\left\{\prod_{\substack{k=1\\k\neq j}}^{M}\sum_{n=1}^{\infty}\left\{\frac{a_kz_k}{\mu_k-\mu_j}\right\}^{n-1}\right\}\\&=\frac{\exp(-\mu_jt)}{\prod_{1\le i\neq j\le M}(\mu_i-\mu_j-a_iz_i)}\end{aligned}$$

and the result follows.

Since the infinite sums enabled the sum and product to be interchanged in the proof, a closed form solution could be found without recourse to normalising constants as in Lemma 1. The corresponding density is now straightforward and we have the following result

Theorem 3.

$$\sum_{n\in T}D(n,t)\prod_{i=1}^{M}a_i^{n_i-1}=\sum_{j=1}^{M}\frac{\exp((a_j-\mu_j)t)}{\prod_{1\le i\neq j\le M}(\mu_i-\mu_j-a_i+a_j)}.$$

Proof.

$$\sum_{n\in T}D(n,t)\prod_{i=1}^{M}a_i^{n_i-1}=\sum_{j=1}^{M}K_j(1,\cdots,1).$$

In a Markov process, the holding times of successive states entered by the process are independent so that the distribution of the time to pass through a given sequence of states is the convolution of state holding time distributions. Since these are negative exponential, the Laplace transform of the passage-time distribution for any given sequence takes the form of a product which can be inverted by Theorem 1. Moreover, certain weighted sums of such convolutions can be inverted analytically via Theorems 2 and 3. This observation is applied to a class of closed queueing networks in the next section to obtain the distribution of cycle time.

3. Passage times in overtake-free queueing networks

We now consider cycle times in tree-structured, overtake-free, closed queueing networks. Such networks are defined recursively and consist of:

- a linear *trunk* segment containing one or more servers, the first being called the *root* server
- a number (greater than or equal to zero) of disjoint *subtrees* of the same structure, to which the last server in the trunk is connected.

The *leaf* servers are those from which departing customers next visit the root server. We assume all servers have exponential service times with constant rates and first-come first-served queueing discipline. Thus in a tree-structured network a customer's passage from the root to a leaf cannot be influenced by any other customer behind him on his chosen path (or on other paths): i.e. such networks have the no-overtaking property (Walrand and Varaiya (1980)), and if paths are restricted to start at one given server (here the root), they define the most general class for which it holds.

Cycle time is defined to be the time elapsed between the arrival at the root of a special customer — called the 'tagged' customer — and his departure from a leaf. Because of the no-overtaking property and the fact that service rates are constant, cycle time is the same as the time elapsed between the tagged customer arriving at the root and leaving the corresponding open network in which departures from leaf servers leave the network. This open network has no external arrivals and so is transient, and we may compute the distribution of the tagged customer's sojourn time in it given the same initial state distribution as would be encountered in the closed network.

We consider a closed tree-structured network A of M servers, with root server numbered 1 and we use the following notation:

- μ_i is the constant service rate of server i $(1 \leqq i \leqq M)$
- e_i is the visitation rate of server i $(1 \leqq i \leqq M)$, defined as a non-zero solution of the equations

$$e_i = \sum_{j=1}^{M} e_j p_{ji} \qquad (1 \leqq i \leqq M)$$

- $S(N) = \{\boldsymbol{n} \mid \Sigma_{i=1}^{M} n_i = N;\ n_i \geqq 0,\ 1 \leqq i \leqq M\}$ is the state space of A
- $G(N) = \Sigma_{\boldsymbol{n} \in S(N)} (\mu_1/e_1)\Pi_{i=1}^{M} (e_i/\mu_i)^{n_i}$ for $\boldsymbol{n} \in S(N)$, $n_1 > 0$.

Thus $G(N)$ is equal to the normalising constant for the stationary state space probabilities of A when its population is $N - 1$, i.e. for $S(N - 1)$.

Now let Z denote the set of all paths through the network A, i.e. sequences of servers entered in passage through A. Then for all $\boldsymbol{z} = (z_1, \cdots, z_k) \in Z$, $z_1 = 1$, z_k is a leaf server and the order of Z is the number of leaf servers since there is only one path from the root to a given leaf in a tree. The probability of choosing path $\boldsymbol{z}$ is equal to the product of the routing probabilities between successive component centres in $\boldsymbol{z}$. The Laplace transform of cycle time distribution is given by the following lemma, obtained in different forms independently by various authors, e.g. Daduna (1982), Kelly and Pollett (1983), Harrison (1984).

Lemma 3. For the tree-structured network A, the Laplace transform of cycle time distribution, conditional on choice of path $z \in Z$ is

$$L(s \mid z) = \frac{1}{G(N)} \sum_{n \in S(N-1)} \prod_{i=1}^{M} \left(\frac{e_i}{\mu_i}\right)^{n_i} \prod_{j=1}^{|z|} \left\{\frac{\mu_{z_j}}{s + \mu_{z_j}}\right\}^{n_{z_j}+1}$$

where $|z|$ is the number of servers in path z.

We can now use Theorem 2 and Lemma 3 to obtain an expression for the density function of cycle time in tree-structured networks, giving the following result.

Theorem 4. If the centres in path $z \in Z$ in the tree-structured network A have distinct service rates, the cycle time density function, conditional on the choice of path $z = (1, 2, \cdots, m)$, is

$$\frac{\prod_{i=1}^{m} \mu_i}{G(N)} \sum_{p=0}^{N-1} G_m(N-p) \sum_{j=1}^{m} \frac{\exp(-\mu_j t)}{\prod_{1 \leqq i \neq j \leqq m} (\mu_i - \mu_j)} \sum_{i=0}^{p} \frac{(e_j t)^{p-i}}{(p-i)!} \sum_{\substack{n \in S_{m,m+i} \\ n_j = 1}} \prod_{\substack{k=1 \\ k \neq j}}^{m} \left[\frac{(e_k - e_j)}{\mu_k - \mu_j}\right]^{n_k - 1}$$

where for $k > 0$, $G_m(k)$ is the normalising constant for the network A with servers $1, 2, \cdots, m$ removed and population $k - 1$, defined by

$$G_m(k) = \sum_{\substack{\Sigma_{i=m+1}^{M} n_i = k-1 \\ n_i \geqq 0}} \prod_{i=m+1}^{M} \left[\frac{e_i}{\mu_i}\right]^{n_i}.$$

Proof. We partition the sum over $S(N-1)$ according to the total number of customers, p, at servers in the path z. Then we have:

$$L(s \mid z) = \frac{1}{G(N)} \sum_{p=0}^{N-1} \sum_{\substack{\Sigma_{i=m+1}^{M} n_i = N-p-1 \\ n_i \geqq 0}} \prod_{i=m+1}^{M} \left[\frac{e_i}{\mu_i}\right]^{n_i} \sum_{\substack{\Sigma_{i=1}^{m} n_i = p \\ n_i \geqq 0}} \prod_{i=1}^{m} \left[\frac{e_i}{\mu_i}\right]^{n_i} \prod_{i=1}^{m} \left[\frac{\mu_i}{s + \mu_i}\right]^{n_i + 1}.$$

Changing the last domain of summation and rearranging then gives:

$$= \frac{1}{G(N)} \sum_{p=0}^{N-1} G_m(N-p) \sum_{\substack{\Sigma_{i=1}^{m} n_i = m+p \\ n_i \geqq 1}} \prod_{i=1}^{m} \mu_i \frac{e_i^{n_i - 1}}{(s + \mu_i)^{n_i}}.$$

The result then follows by inverting the summation of the products using Theorem 2 with $M = m$, $N = m + p$ and $a_i = e_i$ $(1 \leqq i \leqq m)$.

The summations over $S_{m,m+i}$ are just normalising constants that may be computed efficiently along with the $G_m(N-p)$ and $G(N)$ by Buzen's algorithm (Buzen (1973)); see Corollary 1 to Lemma 1. Moreover, if the ratios $(e_i - e_j) : (\mu_i - \mu_j)$ are distinct for each j $(1 \leqq i \neq j \leqq m)$, further simplification is possible via Corollary 2 to that lemma. However, the explicit form is most convenient for relatively simple networks. Finally, the expression given in Theorem 4 simplifies greatly when the network is cyclic, reducing to the result obtained in Harrison (1984).

Corollary to Theorem 4. For a *cyclic* network of M exponential servers and population N, cycle time distribution is

$$\frac{\left[\prod_{i=1}^{M} \mu_i\right] t^{N-1}}{G(N)(N-1)!} \sum_{j=1}^{M} \frac{\exp(-\mu_j t)}{\prod_{1 \leqq i \neq j \leqq M} (\mu_i - \mu_j)}.$$

Proof. Set $e_1 = \cdots = e_M = 1$ in the theorem, so that all terms are zero in the right-most sum except when $n_k = 1$ for all k, i.e. when $i = 0$. Finally, note there is only one partition of the state space, namely the one with all $N - 1$ customers at the servers $1, \cdots, M$. Thus we have $G_M(n) = 1$ if $n = 1$ and $= 0$ if $n > 1$, so that only terms with $p = N - 1$ give a non-zero contribution.

An alternative, direct proof uses the observation made after Theorem 2 in the previous section, after which the result follows trivially.

4. Conclusion

This paper has derived the density function that has for its Laplace transform an arbitrary finite product of negative exponential transforms. In other words, we have obtained an expression for an arbitrary convolution of negative exponentials by analytical inversion of its Laplace transform. The main application seen for this result is to obtain the distribution of the time delays incurred by a Markov process undergoing a given sequence of state transitions or a random choice of sequences with known probabilities. The inversion was obtained both in explicit form and as a recurrence relation, the latter leading to a further recurrence for certain geometrically weighted sums of transforms which could be solved. This result was then used to obtain an expression for the density of cycle times in closed overtake-free queueing networks for the first time. Previously the general result had only been obtained as a Laplace transform, the density function being found only for the special case of cyclic networks. Clearly, it would be desirable to obtain expressions for many other types of passage time between states in a stochastic process, and the present analysis provides a useful starting point. One immediate generalisation of our approach is to use weights of the form $\prod_{i=1}^{M} \prod_{j=1}^{n_i-1} a_i(j)$ in the sums over the state spaces S_{MN}, T_M — i.e. to replace the constants a_i with state-dependent quantities, as arise in queues with variable service rates, for example. We can define generating functions analogous to those of Lemmas 1 and 2, and obtain in exactly the same way the recurrence

$$\sum_{\boldsymbol{n} \in S_N} \left\{ \sum_{k=1}^{M} \frac{n_k - 1}{a_k(n_k - 1) z_k} \right\} D_j(\boldsymbol{n}, t) \prod_{i=1}^{M} \prod_{j=1}^{n_i - 1} a_i(j) z_i = t H_{N-M-1}(\boldsymbol{z}) \qquad (N > M)$$

in the former case, with a similar result for the second. Now, if $a_k(j) = j \cdot \alpha_k$ for constant α_k $(1 \leqq k \leqq M, j \geqq 1)$, this simplifies to:

$$\left[\sum_{k=1}^{M} \frac{1}{\alpha_k z_k}\right] H_n = tH_{n-1} \qquad (n > 0)$$

which is easily solved. Similarly, if $a_k(j) = \alpha_k/(j-2)$ for constant α_k we obtain:

$$\sum_{k=1}^{M} \frac{z_k}{\alpha_k} \frac{\partial^2 H_n}{\partial z_k^2} = tH_{n-1} \qquad (n > 0).$$

In general, if $j/a_k(j)$ is any polynomial in j, we obtain a linear p.d.e. of order equal to the degree of that polynomial.

However, it must be remembered that to obtain passage-time densities in an arbitrary queueing network involves a considerable extension of the state space in order to keep track of the tagged customer, see for example Iglehart and Shedler (1978); even deriving the Laplace transform in a simple, tractable form remains an open question. This has led to the development of approximate methods, such as Harrison (1986), Hohl and Kuehn (1987), but the results of these too are given in terms of Laplace transforms to which the present approach for inversion may be applied.

References

ABATE, T. AND WHITT, W. (1988) Seeing through the Laplace curtain: numerical and approximate methods for Laplace transform inversion. Tutorial at *SIGMETRICS 1988, Santa Fe.*

BUZEN, J. P. (1973) Computational algorithms for closed queueing networks with exponential servers. *Comm. Assoc. Comput. Mach.* **16**, 527–531.

DADUNA, H. (1982) Passage times for overtake-free paths in Gordon–Newell networks. *Adv. Appl. Prob.* **14**, 672–686.

DAVIES, B. AND MARTIN, B. (1979) Numerical inversion of the Laplace transform: a survey and comparison of methods. *J. Comp. Phys.* **33**, 1–32.

GORDON, W. J. AND NEWELL, G. F. (1967) Closed queueing systems with exponential servers. *Operat. Res.* **15**, 254–265.

HARRISON, J. M. (1985) *Brownian Motion and Stochastic Flow Systems*. Wiley, New York.

HARRISON, P. G. (1984) A note on cycle times in tree-like queueing networks. *Adv. Appl. Prob.* **16**, 216–219.

HARRISON, P. G. (1986) An enhanced approximation by pair-wise analysis of servers for time delay distributions in queueing networks. *IEEE Trans. Comp.* **35**, 54–61.

HOHL, S. D. AND KUEHN, P. J. (1987) Approximate analysis of flow and cycle times in queuing networks. In *Proc. 3rd Int. Conf. on Data Communication Systems and their Performance, Rio de Janeiro*. North-Holland, Amsterdam.

IGLEHART, D. L. AND SHEDLER, G. S. (1978) Regenerative simulation of response times in networks of queues. *J. Assoc. Comput. Mach.* **25**, 449–461.

KELLY, F. P. AND POLLETT, P. K. (1983) Sojourn times in closed queueing networks. *Adv. Appl. Prob.* **15**, 638–656.

LAVENBERG, S. AND REISER, M. (1980) Stationary state probabilities at arrival instants for closed queueing networks with multiple types of customers. *J. Appl. Prob.* **17**, 1048–1061.

SCHASSBERGER, R. AND DADUNA, H. (1983) The time for a round trip in a cycle of exponential queues. *J. Assoc. Comput. Mach.* **30**, 146–150.

WALRAND, J. AND VARAIYA, P. (1980) Sojourn times and the overtaking condition in Jacksonian networks. *Adv. Appl. Prob.* **12**, 1000–1018.