Mean-field models for interacting battery-powered devices
(Mean field for interacting fluid models)

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Mean-field analysis for CTMCs

$N$ interacting agents each with $D$ local states

$N \rightarrow \infty$

System of $D$ ODEs (independent of $N$)
Simple example: peer-to-peer software update
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\[ \dot{v}_y(t) = -\rho v_y(t) - \gamma v_y(t) - \frac{\beta v_y(t)v_u(t)}{N} + \lambda v_x(t) \]
Simple example: peer-to-peer software update

\[ \dot{v}_y(t) = -\rho v_y(t) - \gamma v_y(t) - \beta \frac{v_y(t)v_u(t)}{N} + \lambda v_x(t) \]

Rate leaving state \( y \)
Simple example: peer-to-peer software update

\[ \dot{v}_y(t) = -\rho v_y(t) - \gamma v_y(t) - \frac{\beta v'_y(t)v_u(t)}{N} + \lambda v_x(t) \]

Rate leaving state \( y \)

Rate entering state \( y \)

Old node

Updated node

\[ 0.9N \times \]

\[ \times 0.1N \]
Simple example: peer-to-peer software update

**Old node**
- \( \text{on}_y \) (on)
- \( \text{off}_x \) (off)
- \( \beta \frac{U}{N} \)
- \( \rho \)
- \( \gamma \)

**Updated node**
- \( \text{on}_u \) (on)
- \( \text{off}_v \) (off)
- \( \lambda \)
- \( \rho \)

\[ 0.9N \times \]

**Graphical Representation**

- Nodes in state \( y \)
- Nodes in state \( z \)
- Nodes in state \( x \)

**Graphs**

- Rescaled component count:
  - Nodes in state \( y \)
  - Nodes in state \( z \)
  - Nodes in state \( x \)

- Rescaled component count:
  - Nodes in state \( u \)
  - Nodes in state \( v \)
Simple example: peer-to-peer software update

$\beta U \over N$

$0.9N \times$

$0.1N \times$

$N = 20$

Nodes in state $y$
Nodes in state $z$
Nodes in state $x$

Nodes in state $u$
Nodes in state $v$

Time, $t$

Rescaled component count
Simple example: peer-to-peer software update

$N = 50$

Old node

Updated node

$0.9N \times \frac{\beta U}{N}$

$\times 0.1N$
Simple example: peer-to-peer software update

\[ 0.9N \times \]

Old node

\[ \gamma \]

\[ \rho \]

\[ on_y \]

\[ off_x \]

\[ on_z \]

Updated node

\[ \lambda \]

\[ \rho \]

\[ on_u \]

\[ off_v \]

\[ \beta U \]

\[ N \]

\[ 0.1N \]

\[ N = 100 \]

\[ \times \]

\[ 3/8 \]
Simple example: peer-to-peer software update

\[ N = 1000 \]
What if we add a battery to each node?

Additionally, may be dependence of transition rates on own battery level and on that of other nodes.
What if we add a battery to each node?

\[ B_t = \begin{cases} \text{on} \quad : \quad \text{state } x \\ \text{off} \quad : \quad \text{state } y \\ \vdots \end{cases} \]

Additionally, may be dependence of transition rates on own battery level and on that of other nodes...
What if we add a battery to each node?

Old node

Updated node

\[
\begin{align*}
\frac{d B_t}{dt} &= \left\{ f_x(B_t) : \text{state } x \right. \\
&\quad \left. f_y(B_t) : \text{state } y \right. \\
&\quad \ldots \\
\end{align*}
\]
What if we add a battery to each node?

Additionally, may be dependence of transition rates on own battery level
What if we add a battery to each node?

![Diagram of node states](image)

- Additionally, may be dependence of transition rates on own battery level
- ... and on that of other nodes
What if we add a battery to each node?

This is a very large Markovian fluid model (next two talks)
What if we add a battery to each node?

- This is a very large *Markovian fluid model* (next two talks)
  - Number of discrete states is $O(D^N)$

\[
\begin{align*}
\frac{\partial(B_t)}{N} & \left\{ 
\begin{array}{l}
f_x(B_t) : \text{state } x \\
f_y(B_t) : \text{state } y \\
\cdots
\end{array}
\right.
\end{align*}
\]

**Old node**
- $\gamma(B_t)$
- $\lambda(B_t)$
- $\rho(B_t)$
- $\rho'(B_t)$
- $\rho(B_t)$

**Updated node**
- $\gamma(B_t)$
- $\lambda(B_t)$
- $\rho(B_t)$
- $\rho'(B_t)$
- $\rho(B_t)$
What if we add a battery to each node?

This is a very large Markovian fluid model (next two talks)
- Number of discrete states is $O(D^N)$
- Number of fluid variables is $N$
- In theory, can be analysed in the transient regime by solving a system of $O(D^N)$ linear PDEs in $N + 1$ variables
What if we add a battery to each node?

This is a very large *Markovian fluid model* (next two talks)
- Number of discrete states is $O(D^N)$
- Number of fluid variables is $N$
- In theory, can be analysed in the transient regime by solving a system of $O(D^N)$ linear PDEs in $N+1$ variables
- Can we do mean-field for this kind of model; $D$ PDEs with one fluid variable?
Consider a simpler (very toy) wireless sensor network model

- Nodes are in one of two discrete states: *active* (a) or *idle* (i)

\[
\begin{align*}
\text{state } i & : 0 < B_t \leq B^* \\
\text{state } a & : B_t > B^*
\end{align*}
\]
Consider a simpler (very toy) wireless sensor network model

- Nodes are in one of two discrete states: *active* (a) or *idle* (i)
- Idle nodes are awaiting stimulus, active nodes are exchanging collected data with a neighbour
Consider a simpler (very toy) wireless sensor network model

Nodes are in one of two discrete states: *active* (a) or *idle* (i)

- Idle nodes are awaiting stimulus, active nodes are exchanging collected data with a neighbour
- Each has own battery, drains at a state-dependent rate
Consider a simpler (very toy) wireless sensor network model

Nodes are in one of two discrete states: active (a) or idle (i)

- Idle nodes are awaiting stimulus, active nodes are exchanging collected data with a neighbour
- Each has own battery, drains at a state-dependent rate
- Threshold control — wireless radios operate at two different power levels: $0 < B_t \leq B^* \ (low)$ or $B_t > B^* \ (high)$
Consider a simpler (very toy) wireless sensor network model

Nodes are in one of two discrete states: active (a) or idle (i)

Idle nodes are awaiting stimulus, active nodes are exchanging collected data with a neighbour

Each has own battery, drains at a state-dependent rate

Threshold control — wireless radios operate at two different power levels: $0 < B_t \leq B^*$ (low) or $B_t > B^*$ (high)

$$r(A_l(t), A_h(t)) := (\mathbf{1}_{0 < B_t \leq B^*} \beta_l + \mathbf{1}_{B_t > B^*} \beta_h) \frac{A_l(t) + A_h(t) - 1}{N}$$
Consider a simpler (very toy) wireless sensor network model

Nodes are in one of two discrete states: *active* (a) or *idle* (i)

- Idle nodes are awaiting stimulus, active nodes are exchanging collected data with a neighbour
- Each has own battery, drains at a state-dependent rate
- Threshold control — wireless radios operate at two different power levels: $0 < B_t \leq B^*$ (low) or $B_t > B^*$ (high)

\[
r(A_l(t), A_h(t)) := 1_{\{0 < B_t \leq B^*\}} \beta_l \frac{\tau_l (A_l(t) - 1) + \tau_h A_h(t)}{N} + 1_{\{B_t > B^*\}} \beta_h \frac{\tau_l A_l(t) + \tau_h (A_h(t) - 1)}{N}
\]

Presumably $\beta_l \leq \beta_h$, $\tau_l \leq \tau_h$
Consider a simpler (very toy) wireless sensor network model

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- Idle nodes are awaiting stimulus, active nodes are exchanging collected data with a neighbour
- Each has own battery, drains at a state-dependent rate
- Threshold control — wireless radios operate at two different power levels: $0 < B_t \leq B^*$ (*low*) or $B_t > B^*$ (*high*)

\[
\begin{align*}
\frac{dB_t}{dt} &= \begin{cases} 
\gamma_i : \text{state } i, 0 < B_t \\
\gamma_l : \text{state } a, 0 < B_t \leq B^* \\
\gamma_h : \text{state } a, B_t > B^* \\
0 : \text{otherwise}
\end{cases}
\end{align*}
\]

\[r(A_l(t), A_h(t)) := 1_{\{0 < B_t \leq B^*\}} \beta_l \frac{\tau_l (A_l(t) - 1) + \tau_h A_h(t)}{N} + 1_{\{B_t > B^*\}} \beta_h \frac{\tau_l A_l(t) + \tau_h (A_h(t) - 1)}{N}\]

Presumably $\beta_l \leq \beta_h$, $\tau_l \leq \tau_h$ and $\gamma_h \leq \gamma_l \leq \gamma_i \leq 0$
Heuristic derivation of mean-field PDEs

\[ r(A_i(t), A_h(t)) := 1\{0 < B_t \leq B^*\} \beta_l \frac{\tau_l(A_i(t) - 1) + \tau_h A_h(t)}{N} + 1\{B_t > B^*\} \beta_h \frac{\tau_l A_i(t) + \tau_h (A_h(t) - 1)}{N} \]

\[ \frac{dB_t}{dt} = \begin{cases} 
\gamma_i : \text{state } i, \ 0 < B_t \\
\gamma_l : \text{state } a, \ 0 < B_t \leq B^* \\
\gamma_h : \text{state } a, \ B_t > B^* \\
0 : \text{otherwise}
\end{cases} \]

- \( F_a(t, z), F_i(t, z) \): proportion of nodes in a or i, battery \( \leq z \)
Heuristic derivation of mean-field PDEs

\[ r(A_i(t), A_h(t)) := 1_{\{0 < B_t \leq B^*\}} \beta_I \frac{\tau(I(A_i(t)-1) + \tau h A_h(t))}{N} + 1_{\{B_t > B^*\}} \beta_h \frac{\tau_I(A_i(t) + \tau h (A_h(t)-1))}{N} \]

\[ dB_t = \begin{cases} 
\gamma_i: \text{state i, } 0 < B_t \\
\gamma_I: \text{state a, } 0 < B_t \leq B^* \\
\gamma_h: \text{state a, } B_t > B^* \\
0: \text{otherwise}
\end{cases} \]

\[ f_a(t, z) := \frac{\partial}{\partial z} F_a(t, z), \quad f_i(t, z) := \frac{\partial}{\partial z} F_i(t, z) \text{ for } z \in (0, 1] \]
Heuristic derivation of mean-field PDEs

\[ r(A_l(t), A_h(t)) := 1_{\{0 < B_t \leq B^*\}} \beta_l \frac{\tau_l (A_l(t) - 1) + \tau_h A_h(t)}{N} + 1_{\{B_t > B^*\}} \beta_h \frac{\tau_l A_l(t) + \tau_h (A_h(t) - 1)}{N} \]

\[ 1_{\{B_t > 0\}} \lambda r(A_l(t), A_h(t)) \]

\[ \frac{dB_t}{dt} = \begin{cases} 
\gamma_i : \text{state } i, 0 < B_t \\
\gamma_l : \text{state } a, 0 < B_t \leq B^* \\
\gamma_h : \text{state } a, B_t > B^* \\
0 : \text{otherwise}
\end{cases} \]

\[ f_a(t, z) := \frac{\partial}{\partial z} F_a(t, z), \quad f_i(t, z) := \frac{\partial}{\partial z} F_i(t, z) \text{ for } z \in (0, 1) \]

\[ f_a(t + \delta t, z) \approx f_a(t, z) + \left( f_a(t, z) + [1_{\{z \leq B^*\}} \gamma_l + 1_{\{z > B^*\}} \gamma_h] \delta t \right) - f_a(t, z) \]

\[ + \lambda \delta t f_i(t, z) \]

Discharging of batteries in \([t, t+\delta t]\)

Discrete transitions \(i \rightarrow a\) in \([t, t+\delta t]\)

\[ -(1_{\{z \leq B^*\}} \beta_l + 1_{\{z > B^*\}} \beta_h) \delta t f_a(t, z) \left( \tau_l \int_0^{B^*} f_a(t, v) \, dv + \tau_h \int_{B^*}^{1} f_a(t, v) \, dv \right) + o(\delta t) \]

Discrete transitions \(a \rightarrow i\) in \([t, t+\delta t]\)
Heuristic derivation of mean-field PDEs

\[ r(A_l(t), A_h(t)) := 1_{\{0 < B_t \leq B^*\}} \beta_l \frac{\tau_l (A_l(t) - 1) + \tau_h A_h(t)}{N} + 1_{\{B_t > B^*\}} \beta_h \frac{\tau_l A_l(t) + \tau_h (A_h(t) - 1)}{N} \]

\[ dB_t = \begin{cases} \gamma_i : \text{state i, } 0 < B_t \\ \gamma_l : \text{state a, } 0 < B_t \leq B^* \\ \gamma_h : \text{state a, } B_t > B^* \\ 0 : \text{otherwise} \end{cases} \]

\[ f_a(t, z) := \frac{\partial}{\partial z} F_a(t, z), \quad f_i(t, z) := \frac{\partial}{\partial z} F_i(t, z) \text{ for } z \in (0, 1] \]

\[ \frac{\partial f_a(t, z)}{\partial t} - \left(1_{\{z \leq B^*\}} \gamma_l + 1_{\{z > B^*\}} \gamma_h \right) \frac{\partial f_a(t, z)}{\partial z} = \]

\[ \lambda f_i(t, z) - \left(1_{\{z \leq B^*\}} \beta_l + 1_{\{z > B^*\}} \beta_h \right) f_a(t, z) \left( \tau_l \int_0^{B^*} f_a(t, v) \, dv + \tau_h \int_{B^*}^1 f_a(t, v) \, dv \right) \]
Heuristic derivation of mean-field PDEs

\[ r(A_l(t), A_h(t)) := \begin{cases} 0 & \text{if } B_t \leq B^* \\ \beta_l \frac{\tau_l(A_l(t) - 1) + \tau_h A_h(t)}{N} + 1 & \text{if } B_t > B^* \end{cases} \]

\[ \frac{dB_t}{dt} = \begin{cases} \gamma_i : \text{state } i, \; 0 < B_t \\ \gamma_l : \text{state } a, \; 0 < B_t \leq B^* \\ \gamma_h : \text{state } a, \; B_t > B^* \\ 0 : \text{otherwise} \end{cases} \]

- \( e_a(t), e_i(t) \): proportion of nodes in \( a \) or \( i \) with empty battery

\[ e_a(t + \delta t) \approx e_a(t) + \int_0^{\gamma_i \delta t} f_a(t, v) \, dv + o(\delta t) \]

Discharging of batteries in \([t, t+\delta t]\)
Heuristic derivation of mean-field PDEs

\[ r(A_l(t), A_h(t)) := 1_{\{0 < B_t \leq B^*\}} \beta_l \frac{\tau_l(A_l(t) - 1) + \tau_h A_h(t)}{N} + 1_{\{B_t > B^*\}} \beta_h \frac{\tau_l A_l(t) + \tau_h (A_h(t) - 1)}{N} \]

\[ 1_{\{B_t > 0\}} \lambda \rightarrow r(A_l(t), A_h(t)) \rightarrow 1_{\{B_t > 0\}} \lambda \]

\[ \frac{dB_t}{dt} = \begin{cases} \gamma_i : \text{state i, } 0 < B_t \\ \gamma_l : \text{state a, } 0 < B_t \leq B^* \\ \gamma_h : \text{state a, } B_t > B^* \\ 0 : \text{otherwise} \end{cases} \]

► \( e_a(t), e_i(t) \): proportion of nodes in a or i with empty battery

\[ \frac{de_a(t)}{dt} = \gamma_i f_a(t, 0) \]
Heuristic derivation of mean-field PDEs

\[ r(A_l(t), A_h(t)) := \begin{cases} 
\mathbb{1}_{\{0 < B_t \leq B^*\}} \beta_l \frac{\gamma_l(A_l(t)-1)+\gamma_h A_h(t)}{N} + \mathbb{1}_{\{B_t > B^*\}} \beta_h \frac{\gamma_l A_l(t)+\gamma_h (A_h(t)-1)}{N} \\
\end{cases} \]

- System of two non-linear partial (functional) differential equations with ordinary differential equations capturing the mass at zero

\[
\frac{dB_t}{dt} = \begin{cases} 
\gamma_i : \text{state i, } 0 < B_t \\
\gamma_l : \text{state a, } 0 < B_t \leq B^* \\
\gamma_h : \text{state a, } B_t > B^* \\
0 : \text{otherwise}
\end{cases}
\]
Heuristic derivation of mean-field PDEs

\[ r(A_l(t), A_h(t)) := 1_{0 < B_t \leq B^*} \beta_l \frac{\tau_l(A_l(t)-1) + \tau_h A_h(t)}{N} + 1_{B_t > B^*} \beta_h \frac{\tau_l A_l(t) + \tau_h (A_h(t)-1)}{N} \]

\[ 1_{B_t > 0} \lambda r(A_l(t), A_h(t)) \]

\[ \frac{d B_t}{d t} = \begin{cases} 
\gamma_i : \text{state } i, \ 0 < B_t \\
\gamma_l : \text{state } a, \ 0 < B_t \leq B^* \\
\gamma_h : \text{state } a, \ B_t > B^* \\
0 : \text{otherwise} 
\end{cases} \]

- System of two non-linear partial (functional) differential equations with ordinary differential equations capturing the mass at zero
- Specify initial conditions at \( t = 0 \) and also boundary conditions \( f_a(t, 1) = f_i(t, 1) = 0 \) for \( t > 0 \)
- Can be solved inexpensively using standard finite difference techniques
Example solutions

\[ r(A_l(t), A_h(t)) := 1_{\{0 < B_t \leq B^*\}} \beta_l \frac{\tau_l(A_l(t) - 1) + \tau_h A_h(t)}{N} + 1_{\{B_t > B^*\}} \beta_h \frac{\tau_l A_l(t) + \tau_h (A_h(t) - 1)}{N} \]

\[ 1_{\{B_t > 0\}} \lambda r(A_l(t), A_h(t)) \]

\[ \frac{dB_t}{dt} = \begin{cases} 
\gamma_i : \text{state } i, \ 0 < B_t \\
\gamma_l : \text{state a, } 0 < B_t \leq B^* \\
\gamma_h : \text{state a, } B_t > B^* \\
0 : \text{otherwise} 
\end{cases} \]

Battery level, \( z \)  
Time, \( t \)
Example solutions

\[ r(A_l(t), A_h(t)) := 1\{0 < B_t \leq B^*\} \beta_l \frac{\tau_l(A_l(t) - 1) + \tau_h A_h(t)}{N} + 1\{B_t > B^*\} \beta_h \frac{\tau_l A_l(t) + \tau_h (A_h(t) - 1)}{N} \]

\[ dB_t/dt = \begin{cases} 
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\gamma_l : \text{state } a, 0 < B_t \leq B^* \\
\gamma_h : \text{state } a, B_t > B^* \\
0 : \text{otherwise}
\end{cases} \]

\[ N = 10 \]
Example solutions

\[ r(A_l(t), A_h(t)) := 1_{0 < B_t \leq B^*} \beta_l \frac{\tau_l(A_l(t) - 1) + \tau_h A_h(t)}{N} + 1_{B_t > B^*} \beta_h \frac{\tau_l A_l(t) + \tau_h (A_h(t) - 1)}{N} \]

\[ 1_{B_t > 0} \lambda 
\]

\[ B_t \]

\[ \frac{dB_t}{dt} = \begin{cases} 
\gamma_i : \text{state i, } 0 < B_t \\
\gamma_l : \text{state a, } 0 < B_t \leq B^* \\
\gamma_h : \text{state a, } B_t > B^* \\
0 : \text{otherwise} 
\end{cases} \]

\[ N = 20 \]
Example solutions

\[ r(A_l(t), A_h(t)) := 1_{\{0 < B_t \leq B^*\}} \beta_i \frac{\tau_l(A_l(t)-1)}{N} + 1_{\{B_t > B^*\}} \beta_h \frac{\tau_l(A_l(t)+\tau_h(A_h(t)-1)}{N} \]

\[ 1_{\{B_t > 0\}} \lambda \]

\[ r(A_l(t), A_h(t)) \]

\[ B_t \]

\[ \frac{dB_t}{dt} = \begin{cases} 
\gamma_i : \text{state } i, \ 0 < B_t \\
\gamma_l : \text{state a, } 0 < B_t \leq B^* \\
\gamma_h : \text{state a, } B_t > B^* \\
0 : \text{otherwise}
\end{cases} \]

\[ N = 100 \]
Example solutions

\[ r(A_l(t), A_h(t)) := 1_{\{0 < B_t \leq B^*\}} \beta_h \frac{\tau_l(A_l(t) - 1) + \tau_h A_h(t)}{N} + 1_{\{B_t > B^*\}} \beta_h \frac{\tau_l A_l(t) + \tau_h (A_h(t) - 1)}{N} \]

\[ \frac{dB_t}{dt} = \begin{cases} 
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\gamma_l : \text{state a, } 0 < B_t \leq B^* \\
\gamma_h : \text{state a, } B_t > B^* \\
0 : \text{otherwise}
\end{cases} \]

\[ N = 1000 \]
Example solutions

\[ r(A_i(t), A_h(t)) := 1_{\{0 < B_t \leq B^*\}} \beta_l \frac{\tau_l(A_i(t) - 1) + \tau_h A_h(t)}{N} + 1_{\{B_t > B^*\}} \beta_h \frac{\tau_l A_i(t) + \tau_h (A_h(t) - 1)}{N} \]

Compared to single trace

<table>
<thead>
<tr>
<th>Pop. size</th>
<th>Avg. error</th>
<th>Max. error</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N = 10)</td>
<td>0.1425</td>
<td>0.3540</td>
</tr>
<tr>
<td>(N = 100)</td>
<td>0.0264</td>
<td>0.1212</td>
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<tr>
<td>(N = 1000)</td>
<td>0.0138</td>
<td>0.0345</td>
</tr>
<tr>
<td>(N = 10000)</td>
<td>0.0042</td>
<td>0.0336</td>
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</tbody>
</table>

Compared to mean of 1000 traces

<table>
<thead>
<tr>
<th>Pop. size</th>
<th>Avg. error</th>
<th>Max. error</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N = 10)</td>
<td>0.0307</td>
<td>0.0577</td>
</tr>
<tr>
<td>(N = 100)</td>
<td>0.0042</td>
<td>0.0234</td>
</tr>
<tr>
<td>(N = 1000)</td>
<td>Simulation too costly</td>
<td></td>
</tr>
<tr>
<td>(N = 10000)</td>
<td>Simulation too costly</td>
<td></td>
</tr>
</tbody>
</table>
Future directions

- Stability conditions for steady-state mean-field convergence
Future directions

- Stability conditions for steady-state mean-field convergence
- Mean field for second-order fluid models
Thank you, questions?