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Computing optimal multi-currency mean–variance portfolios

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Abstract

The paper provides an exact and computable solution to the general quadratic mean–variance problem with exchange rate and return uncertainties. Multiple currencies are considered, with multiple investments in each currency. The classical mean–variance formulation is augmented with currency risk. The overall risk is hence the multiplicative effect of return risks, within a currency, and the currency risk. Currency risk may be interpreted in terms of, though not restricted to, past forecast errors.

We provide a framework within a general structure for return and exchange rate forecasts and risks. We show that the quadratic mean–variance structure is maintained and provide solutions to computational questions in the evaluation of the optimal portfolio. By extending conventional analysis, we arrive at a surprisingly simple solution which may be a useful addition to the arsenal of finance techniques.

Key words: Currency risk; Portfolio optimization; Quadratic mean–variance analysis; International diversification
JEL classification: C61; C63; D81; G11; G15

1. Introduction

In this paper, we consider the extension of the classical quadratic mean–variance portfolio optimization problem (Markowitz, 1959; Sharpe, 1970) to portfolios in different currencies. The importance of international diversification is well recognised in the literature (e.g., Elton and Gruber, 1991; French and Poterba, 1991; Grauer and Hakanson, 1987; Hull and White, 1987; Stulz, 1981; Uppal, 1993). Some of these studies consider currency hedging

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(e.g., Adler and Jorion, 1992; Kritzman, 1992), whereas others discuss approximate solutions. Solnik (1974) formulates the effect of the nonlinearity arising from the return and exchange uncertainties but only considers hedging to take account of currency risk. Makin (1978) discusses the exchange rate as a separate problem. The main issue arises from the product of two uncertain variables since, as Adler and Dumas (1983) point out, the product of two normal variates is not necessarily normal. Adler and Dumas (1983) consider a log-linear objective and a combination of logarithmic and hedge portfolios.

This paper essentially provides a means for computing the mean and variance of a product of two random variables. The mean and variance of a quadratic function of these variables are considered. The quadratic is essentially a weighted inner (scalar) product of an augmented vector of the two random variables. An exact, and computable, solution to the full mean–variance problem is given below.

Consider the return vector $r^i \in \mathbb{R}^{n^i}$ on n^i investments in currency i,

$$r^i = \mathrm{E}[r^i] + \rho^i,$$

where E denotes expectation and $\rho^i \sim \mathcal{N}(0, \Lambda^i_{\rho\rho})$, with $\Lambda^i_{\rho\rho} = \mathrm{E}[(\rho^i)(\rho^i)^T]$. We have multiple currencies $i = 1, \ldots, \mathscr{I}$. To introduce the portfolio problem in a single currency, let $\omega \in \mathbb{R}^{n^i}$ denote the weights to be attached to each investment. Let $\alpha \in [0,1]$ be a fixed scalar and let $\langle u, v\rangle$ denote the scalar (inner) product $u^T v$. An optimal mean–variance portfolio in currency i is given by

$$\max\{\alpha\langle \mathrm{E}[r^i], \omega\rangle - (1-\alpha)\langle \omega, \Lambda^i_{\rho\rho}\omega\rangle \mid \langle \mathit{1}, \omega\rangle = 1, \omega \geq 0\},$$

where $\mathit{1}$ is the vector with all unit elements. The discussion below does not depend on the nature of the constraints on ω.

Consider the exchange rate, e^i, associated with currency i,

$$e^i = \mathrm{E}[e^i] + \varepsilon^i,$$

where $\varepsilon^i \sim \mathcal{N}(0, \Lambda^i_{\varepsilon\varepsilon})$ is the random error between the exchange rate and its forecast. Clearly, the *exchange rate corresponding to the base currency is unity with certainty*. This does not alter our discussion but needs to be taken account of during computation. The return on investments in currency i is converted to the base currency by evaluating $e^i \times r^i$.

Exchange rates are notoriously difficult to model. However, forecasts are available based on macroeconomic analysis. In its simplest form, $\mathrm{E}[e^i]$ can be interpreted as such a forecast. The error between past forecasts and the actual exchange rate would be characterized by ε^i. The historical value of $\Lambda^i_{\varepsilon\varepsilon}$ can thus be simply evaluated and, along with the forecast $\mathrm{E}[e^i]$, used in the determination of the mean–variance portfolio.

Let r and ρ denote the $(\sum_{i=1}^{\mathscr{I}} n^i)$-dimensional return vectors for all investments in all currencies and the corresponding uncertainties, respectively. Hence, we have

$$r = \mathrm{E}[r] + \rho\,. \tag{1.1}$$

Let e denote the $\mathscr{I}$-dimensional vector of exchange rates,

$$e = \mathrm{E}[e] + \varepsilon\,, \tag{1.2}$$

with

$$\beta = \begin{bmatrix} \rho \\ \varepsilon \end{bmatrix} \sim \mathcal{N}(0, \Lambda) \tag{1.3}$$

and

$$\Lambda = \begin{bmatrix} \Lambda_{\rho\rho} & \Lambda_{\rho\varepsilon} \\ \Lambda_{\varepsilon\rho} & \Lambda_{\varepsilon\varepsilon} \end{bmatrix} = \begin{bmatrix} \mathrm{E}(\rho\rho^T) & \mathrm{E}(\rho\varepsilon^T) \\ \mathrm{E}(\varepsilon\rho^T) & \mathrm{E}(\varepsilon\varepsilon^T) \end{bmatrix}.$$

Most approaches for modelling returns provide an estimate of $\Lambda_{\rho\rho}$ based on historical data. We have already considered the evaluation of $\Lambda_{\varepsilon\varepsilon}$ based on the error between past forecasts and historical exchange rates. $\Lambda_{\varepsilon\rho}$ can be similarly evaluated based on the error between past forecasts of returns, exchange rates, and their respective past actual values.

In Section 2, we extend the traditional mean–variance portfolio optimization problem to investments made in different currencies that take account of currency risk. In Section 3, we discuss the relationships for computational implementation.

2. Multi-currency mean–variance portfolios

Each element of r is associated with an exchange rate. To introduce the multi-currency investment portfolio, we define an operation that assigns an exchange rate to a particular investment. Let the matrix

$$\mathscr{O} \in \mathbb{R}^{(\sum_{i=1}^{\mathscr{I}} n^i)} \times \mathbb{R}^{\mathscr{I}}$$

denote the operator that assigns the exchange rate e^i to an investment in r. Let o_{ij} be the ijth element of $\mathscr{O}$, then we have

$$o_{ij} = \begin{cases} 1 & \text{if } i\text{th investment is in the } j\text{th currency,} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\mathscr{C} = I$ if there is only one investment in each currency. In general, the exchange rate corresponding to each element in the vector r is given by $\mathscr{C}e$. Let

$$\operatorname{diag}[\mathscr{C}e] \in \mathbb{R}^{(\sum_{i=1}^{\mathscr{I}} n^i)} \times \mathbb{R}^{(\sum_{i=1}^{\mathscr{I}} n^i)}$$

be the diagonal matrix whose jth diagonal element is given by the jth element of $\mathscr{C}e$. Then the return in the base currency is given by

$$R = [\operatorname{diag}[\mathscr{C}e]]\, r \in \mathbb{R}^{(\sum_{i=1}^{\mathscr{I}} n^i)}. \tag{2.1}$$

Let $\omega \in \mathbb{R}^{(\sum_{i=1}^{\mathscr{I}} n^i)}$ denote the portfolio weights for the multi-currency problem. Then the total return on the portfolio is given by

$$\langle R, \omega \rangle = \langle r, [\operatorname{diag}[\omega]]\, \mathscr{C}e \rangle. \tag{2.2}$$

Proposition 1. Let r and e be given by (1.1) and (1.2). We then have

$$\mathrm{E}[\langle R, \omega \rangle] = \langle \mathrm{E}[r], [\operatorname{diag}[\omega]]\, \mathscr{C}\mathrm{E}[e] \rangle + \tfrac{1}{2}\operatorname{trace}(\Lambda\Omega(\omega)) \tag{2.3a}$$

for

$$\Omega(\omega) = \begin{bmatrix} 0 & \operatorname{diag}[\omega]\, \mathscr{C} \\ \mathscr{C}^T \operatorname{diag}[\omega] & 0 \end{bmatrix}. \tag{2.3b}$$

Proof. Using (1.1)–(1.2), we can write (2.2) as

$$\begin{aligned} \mathrm{E}[\langle R, \omega \rangle] &= \mathrm{E}[\langle \mathrm{E}[r] + \rho, [\operatorname{diag}[\omega]]\, \mathscr{C}[\mathrm{E}[e] + \varepsilon] \rangle] \\ &= \langle \mathrm{E}[r], [\operatorname{diag}[\omega]]\, \mathscr{C}\mathrm{E}[e] \rangle + \mathrm{E}[\langle \rho, [\operatorname{diag}[\omega]]\, \mathscr{C}\varepsilon \rangle]. \end{aligned}$$

Since

$$\mathrm{E}[\langle \beta, \Omega(\omega)\beta \rangle] = \operatorname{trace}(\Lambda\Omega(\omega)),$$

we have the desired result. □

To consider the variance term, we define the matrix

$$\mathscr{M} = \begin{bmatrix} \operatorname{diag}[\mathscr{C}\mathrm{E}[e]] \\ \mathscr{C}^T \operatorname{diag}\mathrm{E}[r] \end{bmatrix} \in \mathbb{R}^{(\mathscr{I} + \sum_{i=1}^{\mathscr{I}} n^i)} \times \mathbb{R}^{(\sum_{i=1}^{\mathscr{I}} n^i)}.$$

The first $(\sum_{i=1}^{\mathscr{I}} n^i)$ rows of $\mathscr{M}$ consist of the diagonal matrix whose diagonal elements are given by $\mathscr{C}\mathrm{E}[e]$. The last $\mathscr{I}$ rows are given by the product of $\mathscr{C}$ with the diagonal matrix whose diagonal elements are given by $\mathrm{E}[r]$.

Proposition 2. Let r and e be given by (1.1) and (1.2). We then have

$$\mathrm{var}[\langle R,\omega\rangle] = \langle \mathcal{M}\omega, \Lambda\mathcal{M}\omega\rangle + \tfrac{1}{2}\,\mathrm{trace}\,(\Lambda\Omega(\omega))^2 . \tag{2.4}$$

Proof. Using (1.1)–(1.2), we can write (2.3) as

$$\begin{aligned}
\mathrm{var}[\langle R,\omega\rangle] &= \mathrm{E}(\langle r, [\mathrm{diag}\,[\omega]]\,\mathcal{C}e\rangle - \mathrm{E}\,[\langle r, [\mathrm{diag}\,[\omega]]\,\mathcal{C}e\rangle])^2 \\
&= \mathrm{E}(\langle r, [\mathrm{diag}\,[\omega]]\,\mathcal{C}e\rangle - \langle \mathrm{E}\,[r], [\mathrm{diag}\,[\omega]]\,\mathcal{C}\mathrm{E}\,[e]\rangle \\
&\quad - \tfrac{1}{2}\,\mathrm{trace}\,(\Lambda\Omega(\omega)))^2 \\
&= \mathrm{E}(\langle \mathrm{E}\,[r] + \rho, [\mathrm{diag}\,[\omega]]\,\mathcal{C}\,[\mathrm{E}\,[e] + \varepsilon]\rangle \\
&\quad - \langle \mathrm{E}\,[r], [\mathrm{diag}\,[\omega]]\,\mathcal{C}\mathrm{E}\,[e]\rangle \\
&\quad - \tfrac{1}{2}\,\mathrm{trace}\,(\Lambda\Omega(\omega)))^2 \\
&= \mathrm{E}(\langle \mathrm{E}\,[r], [\mathrm{diag}\,[\omega]]\,\mathcal{C}\varepsilon\rangle + \langle \rho, [\mathrm{diag}\,[\omega]]\,\mathcal{C}\mathrm{E}\,[e]\rangle \\
&\quad + \langle \rho, [\mathrm{diag}\,[\omega]]\,\mathcal{C}\varepsilon\rangle - \tfrac{1}{2}\,\mathrm{trace}\,(\Lambda\Omega(\omega)))^2 \\
&= \mathrm{E}(\langle \mathcal{M}\omega, \beta\rangle + \tfrac{1}{2}\langle \beta, \Omega(\omega)\beta\rangle - \tfrac{1}{2}\,\mathrm{trace}\,(\Lambda\Omega(\omega)))^2 \\
&= \mathrm{E}(\langle \mathcal{M}\omega, \beta\rangle)^2 + \tfrac{1}{4}\mathrm{E}(\langle \beta, \Omega(\omega)\beta\rangle)^2 \\
&\quad - \tfrac{1}{2}\,\mathrm{trace}\,(\Lambda\Omega(\omega))\,\mathrm{E}(\langle \beta, \Omega(\omega)\beta\rangle) \\
&\quad + \tfrac{1}{4}\,(\mathrm{trace}\,(\Lambda\Omega(\omega)))^2 \\
&= \mathrm{E}(\langle \mathcal{M}\omega, \beta\rangle)^2 + \tfrac{1}{2}\,\mathrm{trace}\,(\Lambda\Omega(\omega))^2,
\end{aligned}$$

where in the last equality we invoke the relationship

$$\mathrm{E}(\langle \beta, \Omega(\omega)\beta\rangle)^2 = (\mathrm{trace}\,(\Lambda\Omega(\omega)))^2 + 2\,\mathrm{trace}\,(\Lambda\Omega(\omega))^2 . \quad \square$$

As the mean of the portfolio return is given by (2.3) and its variance is given by (2.4), the mean–variance optimization problem can now be formulated to take account of currency risk as well. We summarise this result in Theorem 1.

Theorem 1. *Consider the mean–variance portfolio in which the return R is the product of the return in a local currency and the exchange rate, given by (2.1). Then the mean–variance portfolio*

$$\max\{\alpha \mathrm{E}[\langle R, \omega\rangle] - (1-\alpha)\mathrm{var}[\langle R, \omega\rangle] \mid \langle 1, \omega\rangle = 1,\ \omega \geq 0\}$$

is given by

$$\max\{\alpha[\langle \mathrm{E}[r], [\mathrm{diag}[\omega]]\,\mathcal{O}\mathrm{E}[e]\rangle + \tfrac{1}{2}\,\mathrm{trace}(\Lambda\Omega(\omega))]$$

$$-(1-\alpha)[\langle \mathcal{M}\omega, \Lambda\mathcal{M}\omega\rangle + \tfrac{1}{2}\,\mathrm{trace}(\Lambda\Omega(\omega))^2] \mid \langle 1, \omega\rangle = 1,\ \omega \geq 0\}. \quad \square$$

The simpler case of one investment per currency can be obtained by considering $\mathcal{O}$ as an identity matrix. In order to compute optimal portfolios, the differentials of the trace terms in (2.3) and (2.4), with respect to portfolio weights, need to be easily computable. This is discussed in the next section.

3. Computing the optimal portfolio

A computational difficulty in the evaluation of the optimal mean–variance portfolio as a quadratic programming problem (e.g., Gill, Murray, and Wright, 1981; Luenberger, 1984) is the trace terms in (2.3) and (2.4). Although the mean–variance objective, (2.2) and (2.4), is still clearly a quadratic function of the portfolio weights ω, the quadratic objective function can be simplified by an explicit evaluation of the differential of the trace terms with respect to ω. For (2.4), we consider

$$f(\Omega(\omega)) = \mathrm{trace}(\Lambda\Omega(\omega))^2$$

and

$$\frac{\partial f(\Omega(\omega))}{\partial \omega_s} = \sum_{k,l} \frac{\partial f(\Omega(\omega))}{\partial \Omega_{kl}} \cdot \frac{\partial \Omega_{kl}}{\partial \omega_s}, \tag{3.1}$$

where ω_s is the sth element of ω and Ω_{kl} is the klth element of $\Omega(\omega)$. We invoke a result due to Athans (1968) to yield

$$\frac{\partial f}{\partial \Omega} = 2\Lambda\Omega(\omega)\Lambda. \tag{3.2}$$

Evaluating $\partial\Omega_{kl}/\partial\omega_s$ it can be verified from the structure of $\Omega(\omega)$ that only nonzero elements are

$$\frac{\partial\Omega_{s,n+j(s)}}{\partial\omega_s} = \frac{\partial\Omega_{n+j(s),s}}{\partial\omega_s} = 1, \qquad j(s) \in \{1, \dots, \mathscr{I}\},$$

where $j(s)$ is the index of the element of e indicating the exchange rate corresponding to ω_s. Using (3.1)–(3.2) we get

$$\frac{\partial f(\Omega(\omega))}{\partial\omega_s} = 2\left[[\Lambda\Omega(\omega)\Lambda]_{s,n+j(s)} + [\Lambda\Omega(\omega)\Lambda]_{n+j(s),s}\right]$$

$$= 4\,[\Lambda\Omega(\omega)\Lambda]_{s,n+j(s)}\,.$$

The block containing this element is $\Lambda\Omega(\omega)\Lambda$ is

$$[\Lambda_{\rho\varepsilon}\mathcal{O}^T[\mathrm{diag}[\omega]]\Lambda_{\rho\varepsilon} + \Lambda_{\rho\rho}[\mathrm{diag}[\omega]]\mathcal{O}\Lambda_{\varepsilon\varepsilon}]\,.$$

As n can be eliminated from $n + j(s)$ by just considering this block, we have

$$\frac{\partial f(\Omega(\omega))}{\partial\omega_s} = 4\,[\Lambda_{\rho\varepsilon}\mathcal{O}^T[\mathrm{diag}[\omega]]\Lambda_{\rho\varepsilon} + \Lambda_{\rho\rho}[\mathrm{diag}[\omega]]\mathcal{O}\Lambda_{\varepsilon\varepsilon}]_{s,j(s)}\,.$$

Consider the evaluation of the differential

$$\partial\,\mathrm{trace}(\Lambda\Omega(\omega))/\partial\omega_s$$

arising from (2.3). This can be computed using the above discussion and the relationship

$$\frac{\partial\,\mathrm{trace}(\Lambda\Omega(\omega))}{\partial\Omega} = \Lambda\,,$$

also due to Athans (1968).

4. Concluding remarks

We use extensions of conventional techniques to construct the multiple currency version of quadratic mean–variance portfolio optimization. The resulting portfolio takes account of the return risks in investments, in a currency, and the currency risk, with respect to a base currency. The mean–variance portfolio, given by (2.5), ensures a risk-averse policy with respect to both risks.

Thus the formulation provides an exact solution to the problem of uncertainty in investments in multiple currencies. We arrive at a surprisingly simple solution which may be a useful addition to the arsenal of finance techniques.

References

Adler, M. and B. Dumas, 1983, International portfolio choice and corporate finance, Journal of Finance 38, 925–984.

Adler, M. and P. Jorion, 1992, Universal currency hedges for global portfolios, Journal of Portfolio Management 18, 28–35.

Athans, M., 1968, The matrix minimum principle, Information and Control 11, 592–606.

Elton, E.J. and M.J. Gruber, 1991, Modern portfolio theory and investment analysis (Wiley, New York, NY).

French, K. and J.M. Poterba, 1991, Investor disunification and international equity markets, American Economic Review 81, 222–226.

Grauer, F. and N. Hakanson, 1987, Gains from international diversification, 1968–85: Returns to portfolios of stocks and bonds, Journal of Finance 42, 721–738.

Gill, P., W. Murray, and M. Wright, 1981, Practical optimization (Academic Press, New York, NY).

Hull, J. and A. White, 1987, Hedging the risks from writing foregin currency options, Journal of International Money and Finance 6, 131–152.

Kritzman, M., 1992, The optimal currency hedging policy with biased forward rates, Journal of Portfolio Management 19, 94–100.

Luenberger, D.G., 1984, Linear and nonlinear programming (Addison Wesley, Reading, MA).

Makin, J., 1978, Portfolio theory and the problem of exchange risk, Journal of Finance 33, 517–534.

Markowitz, H., 1959, Portfolio selection: Efficient diversification of investments (Wiley, New York, NY).

Sharpe, W., 1970, Portfolio theory and capital markets (McGraw-Hill, New York, NY).

Solnik, B.H., 1974, Equilibrium model of the international market, Journal of Economic Theory 8, 500–524.

Stulz, R.M., 1981, On the effects of barriers to international investment, Journal of Finance 36, 923–934.

Uppal, R., 1993, A general equilibrium model of international portfolio choice, Journal of Finance 48, 529–553.