ELSEVIER

Journal of Economic Dynamics & Control
24 (2000) 1591–1621

JOURNAL OF
Economic
Dynamics
& Control

www.elsevier.nl/locate/econbase

Robust min–max portfolio strategies for rival forecast and risk scenarios

Berç Rustem[a],*, Robin G. Becker[b], Wolfgang Marty[b]

[a]*Department of Computing, Imperial College, 180 Queen's Gate, London SW7 2BZ, UK*
[b]*Warburg Dillon Read, 1 High Timber Street, London EC4V 3SB, UK*

Abstract

We consider an extension of the Markowitz mean–variance optimization framework to multiple return and risk scenarios. It is well known that asset return forecasts and risk estimates are inherently inaccurate. The method proposed provides a means for considering rival representations of the future. The optimal portfolio is computed, simultaneously with the worst case, to take account of all rival scenarios. This is a min-max strategy which is essentially equivalent to a robust pooling of the scenarios. Robustness is ensured by the noninferiority of min–max. For example, a basic worst-case optimal return is guaranteed in view of multiple return scenarios. If robustness happens to have too high a cost, guided by the min–max pooling, it is also possible to explore other pooling alternatives. A min–max algorithm is used to solve the problem and illustrate the robust character of min–max with return and risk scenarios. We study the properties of the min–max risk–return frontier and compare with the potentially suboptimal worst-case where the investment strategy and the worst case are computed separately. © 2000 Elsevier Science B.V. All rights reserved.

JEL classification: C44; C61; C63; G11

Keywords: Risk management; Robust decisions; Guaranteed return; Portfolio optimisation; Multiple scenarios; Worst-case analysis; Min–max decisions

* Corresponding author.

E-mail address: br@doc.ic.ac.uk (B. Rustem).

0165-1889/00/$ - see front matter
PII: S0165-1889(99)00088-3

1. Introduction: The optimal portfolio problem

The success of many financial strategies is measured by comparisons against a variety of benchmarks. Important examples are equity tracking funds. In this paper, we consider a new approach which extends the classical mean–variance framework due to Markowitz to allow for multiple forecast scenarios. The optimal decision is based on a min–max strategy which ensures robustness to scenarios. The optimal portfolio is no longer determined by a single scenario but depends simultaneously on all the scenarios.

Consider the mean–variance framework for given $\alpha \in [0, \infty]$

$$\min_{\omega} \{J_\alpha(\omega) | \omega \in \Omega\}, \tag{1.1}$$

where $J_\alpha(\omega)$ is the quadratic objective function[1]

$$J_\alpha(\omega) = -\langle \mathscr{E}(r), \omega \rangle + \alpha \langle \omega - \bar{\omega}, \Lambda(\omega - \bar{\omega}) \rangle, \tag{1.2}$$

$\mathscr{E}(r) \in \mathbb{R}^n$ is the expected return vector of the set of assets being considered with

$$r = \mathscr{E}(r) + \varepsilon, \tag{1.3}$$

$\varepsilon \sim \mathscr{N}(0, \Lambda)$ is the random error, $\Lambda \in \mathbb{R}^{n \times n}$ is the covariance matrix of the returns, $\omega \in \mathbb{R}^n$ is the portfolio weights to be optimally determined, $\bar{\omega}$ denotes the benchmark weights which ω should follow closely, and Ω is the feasible set of these weights, including the restrictions specified by the investor. This formulation enables the tracking of benchmark $\bar{\omega}$. The error ε can be viewed either as the error between the actual return and its forecast, $\mathscr{E}(r)$ or, the error of the return from its historical mean, when $\mathscr{E}(r)$ is equivalent to this mean. Consequently, Λ can be the covariance of the return forecast errors or the historical covariance of the return. The former seems to be the risk measure more consistent with the return forecasts. The parameter α is the price of risk, interpreted as the shadow price of an associated optimization constrained by the quadratic risk in (1.2). Accordingly, as α increases from zero, so does the emphasis on caution, or risk aversion.

The main difficulty with (1.1) is the importance of the forecast return, $\mathscr{E}(r)$, and risk, Λ, estimates in the determination of the investment strategy. Although it seems natural that these data should be sufficiently precise for the optimal strategy to be useful, the inherent inaccuracy of these estimates is well known in finance. In practice, therefore, the formulation of the optimal portfolio problem (1.1) is an oversimplification. Originating from rival economic and financial

[1] This formulation ensures the simultaneous maximization of expected return and minimization of expected risk.

theories, there exist rival return forecasts purporting to represent the same financial system. The problem of forecasting has been approached through forecast pooling by Fuhrer and Haltmaier (1986), Granger and Newbold (1977), Lawrence et al. (1986) and Makridakis and Winkler (1983). The extension of pooling to optimal policy design is often achieved using stochastic programming approaches by considering the probability of each scenario and evaluating approximate expected values (e.g. Kall and Wallace, 1994; Pardalos and Sandstrom, 1994). An alternative is discussed in Becker et al. (1986) where the robust pooling is computed using a min–max approach. In this paper, we propose a min–max strategy, based on multiple scenarios for these forecasts, to inject robustness in view of such inaccuracies. The *robustness of min–max* stems from its basic property: it is the best decision determined simultaneously with the worst-case scenario. In Section 2, we introduce the strategy and establish its robustness. The rest of this paper is organized as follows: in Section 3, we describe the basic min–max algorithm. In Section 4, we consider variations of the basic strategy to return and risk scenarios. In Section 5, we discuss numerical experiments on optimal bond portfolios. Finally, in Section 6 we present our summary and conclusions.

2. Rival forecast scenarios and robustness of min–max

In the presence of rival forecasts for an investment policy, the investor may wish to take account of *all existing rival scenarios in the design of the optimal policy*. One strategy in such a situation is to adopt the worst-case design problem

$$\min_{\omega} \max_{i} \{J^i_\alpha(\omega) | \omega \in \Omega;\ i = 1, \dots, m^{\text{sce}}\}, \tag{2.1}$$

where there are $i = 1, \dots, m^{\text{sce}}$ scenarios, J^i_α, denoting the objective function for the ith scenario. This is an extension of the intuitive, but potentially *suboptimal*, approach based on determining the optimal strategy corresponding to each scenario and adopting the least damaging strategy in view of all the scenarios (Chow, 1979). Optimality in (2.1) is no longer based on a single scenario, but on all the scenarios simultaneously.

Example. (Rival exchange rate scenarios). Among rival forecast problems, an important class is the exchange rate scenarios arising in multi-currency investments. The rival returns arising from each forecast scenario can be used in the above framework to determine the best decision, robust to differing exchange rate regimes. Suppose $\mathscr{E}(r)$ is the only return scenario in a given currency and there are e^i, $i = 1, \dots, m^{\text{sce}}$, exchange rate scenarios for that currency. Thus, the corresponding overall return scenario is $(e^i \times \mathscr{E}(r))$, $i = 1, \dots, m^{\text{sce}}$.

Problem (2.1) seeks the optimal strategy corresponding to the most adverse circumstance due to choice of scenario. All rival scenarios are assumed to be known. The solution of (2.1) clearly does not provide general insurance against the eventuality that an unknown ($m^{\text{sce}}+1$)st scenario might actually represent the system; it is just a robust strategy against *known* competing 'scenarios'. Nevertheless, it does provide some specific cover for unknown scenarios, as discussed below.[2]

To introduce the basic terminology, let $\hat{J}(\omega)$: $\mathbb{R}^n \to \mathbb{R}^n$ be a twice continuously differentiable function

$$\hat{J}(\omega) = \begin{bmatrix} J_\alpha^1(\omega) \\ \vdots \\ J_\alpha^i(\omega) \\ \vdots \\ J_\alpha^{m^{\text{sce}}}(\omega) \end{bmatrix}.$$

Let $1^\lambda \in \mathbb{R}^{m^{\text{sce}}}$, $1^\omega \in \mathbb{R}^n$ be vectors whose elements are all unity. We define the simplices

$$\mathbb{R}_+^{m^{\text{sce}}} = \{\lambda \in \mathbb{R}^{m^{\text{sce}}} | \langle \lambda, 1^\lambda \rangle = 1; \quad \lambda \geq 0\}, \tag{2.2}$$

and[3]

$$\mathbb{R}_+^n = \{\omega \in \mathbb{R}^n | \langle \omega, 1^\omega \rangle = 1; \quad \omega \geq 0\}. \tag{2.3}$$

In practice, $\Omega \subseteq \mathbb{R}_+^n$. The results below are applicable to a general feasible set for which the first-order optimality conditions of (1.1) are well defined (Rustem, 1998, Chapter 1).

Problem (2.1) is solved by the nonlinear programming problem which optimizes the outcome corresponding to the worst-case scenario. Let v denote the value of the worst-case objective. Thus, v is the worst-case outcome being minimized

$$\min_{\omega, v \in \mathbb{R}^{n+1}} \{v | \hat{J}(\omega) \leq 1^\lambda v; \; \omega \in \Omega\}, \tag{2.4}$$

where $v \in \mathbb{R}^1$.

[2] A general continuum of scenarios addresses the case of uncertain parameters assuming values within a given continuous range. Examples are discussed in Howe et al. (1994,1996), Howe and Rustem (1997) and an algorithm for continuous min–max is given in Rustem and Howe (1999).

[3] Although the restriction $\omega \geq 0$ excludes short sales for simplicity and straightforwad adherence to the original mean–variance formulation, the approach discussed in this paper admits all convex feasible sets, Ω.

It should be noted that min–max is a cautious strategy based on the worst-case scenario. The insurance provided by the strategy may be relaxed by adopting a less extreme formulation based on a dual approach to (2.1) and (2.4) (Rustem, 1987, 1992). Lemmas 1 and 2 below introduce the dual approach and establish an equivalent min–max scenario pooling representation to show the robustness of min–max.

Lemma 1. (Pooling representation). *Problem* (2.1) *is equivalent to the continuous min–max problem*

$$\min_{\omega} \max_{\lambda} \{\langle \lambda, \hat{J}(\omega)\rangle | \lambda \in \mathbb{R}_{+}^{m^{sce}};\ \omega \in \Omega\}. \tag{2.5}$$

Proof. The maximum of m^{sce} numbers is equal to the maximum of their convex combination (Medanic and Andjelic, 1971, 1972; Cohen, 1981). □

Let λ_* be the value of λ that solves (2.5). The first-order optimality conditions of (2.4) and (2.5) show that λ_* is also the shadow price associated with the inequality constraints in (2.4). A feature of (2.5) that makes it preferable to (2.1) is that λ^i can also be interpreted as the importance attached by the investor to scenario i.

The significance of (2.5) is that it represents *the multiple scenario problem as the pooling of the scenarios* with the vector λ. In (2.5), λ is determined to maximize the objective, as the worst- case. An alternative would be the specification of λ by the investor, prior to the solution of the optimization problem. In that case, λ would indicate the prior beliefs of the investor in the scenarios, with (2.5) replaced by

$$\min_{\omega} \{\langle \lambda, \hat{J}(\omega)\rangle | \omega \in \Omega\}. \tag{2.6}$$

This represents a milder, and less robust, alternative to min–max. There may be cases in which the min–max solution λ_* may be too cautious to implement. The investor may then wish to assign a value to λ, in a neighbourhood of λ_*, and consider a more acceptable strategy by minimizing $\langle \lambda, \hat{J}(\omega)\rangle$, with respect to ω, for the given λ. Another interpretation of (2.5) is in terms of the robust character of min–max discussed in Lemma 2 below.

In order to discuss the robustness property explicitly, we need to adopt a reasonably general formulation of the feasible set Ω and then consider the optimality conditions of (2.1), (2.4) and (2.5) (see Rustem, 1998). Let the feasible set be given by

$$\Omega = \{\omega \in \mathbb{R}_{+}^{n} \mid F(\omega) \leq 0\}, \tag{2.7}$$

where F is a vector-valued function whose elements are continuously differentiable. We define the Lagrangian of (2.5), for Ω given by (2.7), by

$$\mathscr{L}(\omega, \lambda, \sigma, \zeta, \theta, \mu, \eta) = \langle \lambda, \hat{J}(\omega) \rangle - \langle \omega, \sigma \rangle + (\langle \omega, 1^{\omega} \rangle - 1)\zeta + \langle F(\omega), \theta \rangle + \langle \lambda, \mu \rangle + (\langle \lambda, 1^{\lambda} \rangle - 1)\eta$$

where $\sigma, \zeta, \theta, \mu, \eta$ are the multipliers of $\omega \geq 0$, $\langle \omega, 1^{\omega} \rangle = 1$, $F(\omega) \leq 0$, $\lambda \geq 0$ and $\langle 1^{\lambda}, \lambda \rangle = 1$, respectively. Let $\omega_*, \lambda_*, \sigma_*, \zeta_*, \theta_*, \mu_*, \eta_*$ solve (2.5). The necessary conditions for optimality of (2.5), for Ω given by (2.7), are

$$\nabla_{\omega} \hat{J}(\omega_*)\lambda_* - \sigma_* + 1^{\omega}\zeta_* + \nabla_{\omega} F(\omega_*)\theta_* = 0, \tag{2.8a}$$

$$\hat{J}(\omega_*) + \mu_* + 1^{\lambda}\eta_* = 0, \tag{2.8b}$$

$$\langle 1^{\omega}, \omega_* \rangle = 1, \quad \omega_* \geq 0, \quad \langle \omega_*, \sigma_* \rangle = 0, \quad \sigma_* \geq 0, \tag{2.8c}$$

$$F(\omega_*) \leq 0, \quad \langle F(\omega_*), \theta_* \rangle = 0, \quad \theta_* \geq 0, \tag{2.8d}$$

$$\langle 1^{\lambda}, \lambda_* \rangle = 1, \quad \lambda_* \geq 0, \tag{2.8e}$$

$$\langle \lambda_*, \mu_* \rangle = 0, \quad \mu_* \geq 0. \tag{2.8f}$$

The min–max strategy ensures a basic guaranteed performance (e.g. return) in view of the multiple scenarios. Its noninferiority, or robustness, ensures that the strategy ensures an optimal guaranteed lower bound on performance. Lemma 2 below is an explicit statement of this property, originally discussed in Rustem (1987). We consider its application to (2.1), (2.4)–(2.5) and discuss the basic guaranteed performance, noninferiority, robustness of min–max.

Lemma 2. (Robustness). *Let*

(i) *Ω be given by* (2.7);
(ii) *there exists a min–max solution to* (2.5), *denoted by* (ω_*, λ_*);
(iii) *$\hat{J}(\omega)$ and $F(\omega)$ be once differentiable at ω_*; and,*
(iv) *strict complementarity hold for $\lambda \geq 0$ at the solution.*

Then, for $i, j, \ell \in \{1, 2, \dots, m^{\text{sce}}\}$, we have

(a) $J^i_{\alpha}(\omega_*) = J^j_{\alpha}(\omega_*)$, $\quad \forall i, j \ (i \neq j)$ $\quad$ iff $\lambda^i_*, \lambda^j_* \in (0, 1)$;
(b) $J^i_{\alpha}(\omega_*) = J^j_{\alpha}(\omega_*) > J^{\ell}_{\alpha}(\omega_*)$, $\quad \forall i, j, \ell (\ell \neq i, j)$ $\quad$ iff $\lambda^{\ell}_* = 0$ and $\lambda^i_*, \lambda^j_* \in (0, 1)$;
(c) $J^i_{\alpha}(\omega_*) > J^j_{\alpha}(\omega_*)$, $\quad \forall j \ (j \neq i)$ $\quad$ iff $\lambda^i_* = 1$;
(d) $J^i_{\alpha}(\omega_*) < J^j_{\alpha}(\omega_*)$, $\quad \forall j \ (j \neq i)$ $\quad$ iff $\lambda^i_* = 0$.

Proof. For (a), necessity can be shown by considering (2.8f), which yields

$$\lambda^i_* \mu^i_* = \lambda^j_* \mu^j_* = 0 \quad \text{for } \lambda^i_*, \lambda^j_* \in (0, 1)$$

and then $\mu^i_* = \mu^j_* = 0$. Using (2.8b) we have $J^i_{\alpha}(\omega_*) = J^j_{\alpha}(\omega_*)$.

Sufficiency is established with $J^i_\alpha(\omega_*) = J^j_\alpha(\omega_*)$ and noting from (2.8b)

$$\langle \lambda_*, \hat{J}(\omega_*) + \mu_* + 1^\lambda \eta_* \rangle = 0.$$

From (2.8e) and (2.8f) we have $\langle 1^\lambda, \lambda_* \rangle = 1$ and $\langle \lambda_*, \mu_* \rangle = 0$ and

$$\eta_* = -\langle \hat{J}(\omega_*), \lambda_* \rangle.$$

Premultiplying equality (2.8b) by 1^λ and using this equality yields

$$0 = \langle 1^\lambda, \hat{J}(\omega_*) \rangle + \langle 1^\lambda, \mu_* \rangle + \langle 1^\lambda, 1^\lambda \rangle \eta_* = \langle 1^\lambda, \mu_* \rangle.$$

By (2.8f), $\mu_* = 0$ and strict complementarity implies that $\lambda^i_*, \lambda^j_* \in (0, 1)$, $\forall i, j$.

To show case (b), consider (2.8f) for $\lambda^i_*, \lambda^j_* \in (0, 1)$, $\lambda^\ell_* = 0$. We have

$$\lambda^i_* \mu^i_* = \lambda^j_* \mu^j_* = \lambda^\ell_* \mu^\ell_* = 0,$$

thence $\mu^i_* = \mu^j_* = 0$ and, by strict complementarity, $\mu^\ell_* > 0$. From (2.8b) we have

$$0 = J^m_\alpha(\omega_*) + \eta_* + \mu^m_*, \quad m = i, j; \tag{2.9a}$$

$$0 = J^\ell_\alpha(\omega_*) + \eta_* + \mu^\ell_*, \tag{2.9b}$$

and thus

$$J^\ell_\alpha(\omega_*) - J^m_\alpha(\omega_*) = -\mu^\ell_* < 0, \quad m = i, j.$$

For sufficiency, let $J^i_\alpha(\omega_*) = J^j_\alpha(\omega_*) > J^\ell_\alpha(\omega_*)$. Combining (2.9) and (2.8f) yields

$$\lambda^\ell_*(J^\ell_\alpha(\omega_*) - J^m_\alpha(\omega_*)) = \lambda^\ell_*(\mu^m_* - \mu^\ell_*) = \lambda^\ell_* \mu^m_* \geq 0.$$

Since $J^\ell_\alpha(\omega_*) - J^m_\alpha(\omega_*) < 0$, we have $\lambda^\ell_* = 0$. With $\lambda^\ell_* = 0$, $\forall \ell$, $J^\ell_\alpha(\omega_*) < J^m_\alpha(\omega_*)$, we can use (a) for those i, j for which $J^i_\alpha(\omega_*) = J^j_\alpha(\omega_*)$ to establish $\mu^i_* = \mu^j_* = 0$. By strict complementarity this implies that $\lambda^i_*, \lambda^j_* \in (0, 1)$.

Case (c) can be established noting that for $\lambda^i_* = 1$, we have $\mu^i_* = 0$, $\lambda^j_* = 0$, $\forall j \neq i$ and, by strict complementarity, $\mu^j_* > 0$. From (2.8b) and (2.8f) we thus obtain

$$J^j_\alpha(\omega_*) - J^i_\alpha(\omega_*) \leq \mu^i_* - \mu^j_* = -\mu^j_* < 0.$$

Conversely, $J^i_\alpha(\omega_*) > J^i_\alpha(\omega_*)$ implies

$$\lambda^j_*(J^j_\alpha(\omega_*) - J^i_\alpha(\omega_*)) = \lambda^j_* \mu^i_* \geq 0$$

and thus $\lambda^j_* = 0$, $\forall j \neq i$. Case (d) is the converse of (c). $\square$

The above result illustrates the way in which λ_* is related to $\hat{J}(\omega_*)$. When some of the elements of λ_* are such that $\lambda^i_* \in (0, 1)$ for some $i \in M \subseteq \{1, 2, \ldots, m^{\text{sce}}\}$, it is shown that the $J^i_\alpha(\omega_*)$, $i \in M$, have the same value. *In this* *case, the optimal policy ω_* yields the same objective function value whichever forecast scenario happens to be realised.* Thus, ω_* is a *robust policy*. The investor is ensured that implementing ω_* will yield an objective function value that is at least as good as the min–max optimum. This noninferiority of ω_* may, on the other hand, amount to a cautious approach with a high cost. The investor can, in such circumstances, use λ_* as a guide and seek in its neighbourhood a slightly less cautious scheme that is more acceptable. Choosing $\lambda = \bar{\lambda}$ from a reasonably close neighbourhood of λ_*, the optimal strategy is based on (2.6). In (2.6), $\bar{\lambda}$ is fixed and represents a pooling of all the scenarios. In general, $\bar{\lambda} \in \mathbb{R}_+^{m^{\text{sce}}}$ can be chosen arbitrarily to reflect the views and expectations of the investor. However, as all expected value optimization needs to be justified in view of the worst-case scenario, a choice in the neighbourhood of λ_* would be desirable.

Example (*Robust pooling*). If the min–max over three functions $J_\alpha^1(\omega)$, $J_\alpha^2(\omega)$ and $J_\alpha^3(\omega)$ is being computed, then, at the solution, $J_\alpha^1 = J_\alpha^2(\omega) > J_\alpha^3$ iff, $\lambda_1, \lambda_2 \in (0, 1)$ and $\lambda_3 = 0$ or $J_\alpha^1 > J_\alpha^2(\omega) \geq J_\alpha^3$ iff $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = 0$. Lemma 2 states this in greater generality. Since λ is chosen to maximize the Lagrangian, the solution can be seen as a robust optimum in the sense of a worst-case design problem.

Example (*Benchmark-relative return formulation*). The min–max framework admits the possibility of benchmark-relative return maximization. Consider the objective function

$$\min_{\omega} \{ - \langle \mathscr{E}(r), \omega - \bar{\omega} \rangle + \alpha \langle \omega - \bar{\omega}, \Lambda(\omega - \bar{\omega}) \rangle | \omega \in \Omega \}, \tag{2.10}$$

which attempts to maximize the benchmark-relative return as opposed to absolute return. This is important when the aim is to explore the possibility of outperforming a benchmark. Unfortunately, as $\langle \mathscr{E}(r), \bar{\omega} \rangle$ is a constant term, solution of (2.10) is identical to the original mean–variance problem in (1.1)–(1.2). This can be verified by comparing the optimality conditions of both problems (see Rustem, 1998). Thus, as the absolute value of portfolio return is already being maximized in (1.1)–(1.2), there is nothing more that can be done to improve the performance further relative to a benchmark.

In the min–max formulation, the above concern about performance can be properly addressed. In general, as different scenarios lead to different portfolio return values, the measurement of the performance of the portfolio return relative to a benchmark would be of interest. In this case, relative return will influence portfolio choice and performance. Consider (2.1) with an objective function augmented by the relative return term corresponding to scenario i, $[\langle \mathscr{E}(r), \bar{\omega} \rangle]^i$,

$$\min_{\omega} \max_{i} \{ J_\alpha^i(\omega) + [\langle \mathscr{E}(r), \bar{\omega} \rangle]^i \, | \omega \in \Omega; \quad i = 1, \ldots, m^{\text{sce}} \}. \tag{2.11}$$

It is straightforward to show that the equivalent nonlinear programming formulation, corresponding to (2.4), is given by

$$\min_{\omega, v \in \mathbb{R}^{n+1}} \{ v \mid \mathscr{g}(\omega) \leq 1^{\lambda} v, \quad \omega \in \Omega \},$$

where

$$\mathscr{g}(\omega) = \begin{bmatrix} J_\alpha^1(\omega) + [\langle \mathscr{E}(r), \bar{\omega} \rangle]^1 \\ \vdots \\ J_\alpha^i(\omega) + [\langle \mathscr{E}(r), \bar{\omega} \rangle]^i \\ \vdots \\ J_\alpha^{m^{\text{sce}}}(\omega) + [\langle \mathscr{E}(r), \bar{\omega} \rangle]^{m^{\text{sce}}} \end{bmatrix}.$$

It can be verified from optimality conditions (2.8) that benchmark-relative return considers the best benchmark-relative portfolio performance for the worst-case scenario.

Consider scenarios defined by rival return estimates $\mathscr{E}(r^i)$; $i = 1, \ldots, m^{\text{sce}}$and the portfolio problem

$$\min_{\omega} \max_{i} \{ -\langle \mathscr{E}(r^i), \omega - \bar{\omega} \rangle + \alpha \langle \omega - \bar{\omega}, \Lambda(\omega - \bar{\omega}) \rangle | \omega \in \Omega,$$
$$i = 1, \ldots, m^{\text{sce}} \}.$$

The solution is the best benchmark-relative performance in view of the worst-case return scenario.

3. The min–max algorithm

Algorithms for solving (2.1) have been considered by a number of authors, including Charalambous and Conn (1978), Coleman (1978), Conn (1979), Demyanov and Maloemzov (1974), Demyanov and Pevnyi (1972), Dutta and Vidyasagar (1977), Goffin et al. (1992), Han (1981), Murray and Overton (1980), Polak et al. (1988). In the constrained case, discussed in some of these studies, global and local convergence rates have not been established (e.g. Coleman, 1978; Dutta and Vidyasagar, 1977). In Rustem (1992,1998), a dual approach to (2.1) is used, along with the primal, to formulate a superlinearly convergent algorithm.

The algorithm for general nonlinear constraints is discussed in detail in Rustem (1992) and Rustem and Nguyen (1998). If the constraints are linear equalities and inequalities, as is mostly the case for optimal portfolio problems in finance, the algorithm simplifies considerably and is relatively easy to implement. In this section, we describe the algorithm for linear constraints which is used in computing the results discussed in this paper. Consider, therefore, the feasible set

$$\Omega = \{\omega \in \mathbb{R}^n_+ | F^{\mathrm{T}}\omega \leq f\}. \tag{3.1}$$

where $F^{\mathrm{T}}\omega \leq f$ is a system of linear inequalities with F a matrix of n rows. We assume that $\Omega \neq \emptyset$.

Let $H(\cdot)$ denote the Hessian of the Lagrangian $\mathscr{L}$, with respect to ω, evaluated at $(\cdot)$. Consider the pooled objective function

$$\mathscr{F}(x, \lambda) = \langle \lambda, \hat{J}(x) \rangle$$

and its linear approximation, with respect to ω, at a point ω_k,

$$\mathscr{F}_k(\omega, \lambda) = \langle \lambda, (\hat{J}(\omega_k) + \nabla \hat{J}(\omega_k)^{\mathrm{T}}(\omega - \omega_k)) \rangle, \tag{3.2}$$

where $\nabla\hat{J}(\omega) \in \mathbb{R}^{n \times m^{\mathrm{sce}}}$ is the matrix

$$\nabla\hat{J}(\omega) = [\nabla J^1_\alpha(\omega) \vdots \cdots \vdots \nabla J^{m^{\mathrm{sce}}}_\alpha(\omega)].$$

We shall sometimes denote $\hat{J}(\omega)$ and $\nabla\hat{J}(\omega)$, evaluated at ω_k, by $\hat{J}_k$ and $\nabla\hat{J}_k$, respectively. Thus, for

$$d = \omega - \omega_k,$$

(3.2) can be written as

$$\mathscr{F}_k(\omega_k + d, \lambda) = \langle \lambda, \hat{J}_k + \nabla\hat{J}^{\mathrm{T}}_k d \rangle.$$

The quadratic objective function used to compute the direction of progress is given by

$$\mathscr{F}_k(\omega_k + d, \lambda) + \tfrac{1}{2}\langle d, \hat{H}_k d \rangle$$

where the matrix $\hat{H}_k$ is a symmetric positive semi-definite approximation to $H(\omega_k)$. The direction of progress at each iteration of the algorithm is determined by the quadratic subproblem

$$\min_{\omega} \max_{\lambda} \{\mathscr{F}_k(\omega_k + d, \lambda) + \tfrac{1}{2}\langle d, \hat{H}_k d \rangle | \omega_k + d \in \Omega; \quad \lambda \in \mathbb{R}^{m^{\mathrm{sce}}}_+\}. \tag{3.3}$$

Since the min–max subproblem is more complex, we also consider the quadratic programming subproblem

$$\min_{\omega, v \in \mathbb{R}^{n+1}} \{v + \tfrac{1}{2}\langle d, \hat{H}_k d \rangle | \omega_k + d \in \Omega; \quad \nabla\hat{J}^{\mathrm{T}}_k d + \hat{J}_k \leq 1^\lambda v\}. \tag{3.4}$$

The two subproblems are equivalent, but (3.4) involves fewer variables. The multipliers associated with $\nabla\hat{J}^{\mathrm{T}}_k d + \hat{J}_k \leq 1^\lambda v$ are the values and the solution of either subproblem satisfies common optimality conditions (Rustem, 1992).

Let the value of (d, λ, v) solving (3.3) and (3.4) be denoted by $(d_k, \lambda_{k+1}, v_{k+1})$. The stepsize along d_k is defined using the equivalent min–max formulation (2.1). Thus, consider the function

$$\psi(\omega) = \max_{i \in \{1,2,\dots,m^{\text{sce}}\}} \{J_\alpha^i(\omega)\}$$

and

$$\psi_k(d) = \max_{i \in \{1,\ 2,\dots,m^{\text{sce}}\}} \{J_\alpha^i(\omega_k) + \langle \nabla J_\alpha^i(\omega_k), d \rangle\}.$$

The stepsize strategy determines τ_k as the largest value of $\tau = (\gamma)^j$, $\gamma \in (0, 1)$, $j = 0$, 1, 2,… such that ω_{k+1} given by

$$\omega_{k+1} = \omega_k + \tau_k d_k \tag{3.5}$$

satisfies the inequality

$$\psi(\omega_{k+1}) - \psi(\omega_k) \leq \rho \tau_k \Phi(d_k), \tag{3.6a}$$

where $\rho \in (0, 1)$ is a given scalar and

$$\Phi(d_k) = \psi_k(d_k) - \psi(\omega_k) + \tfrac{1}{2}\langle d_k, \hat{H}_k d_k \rangle. \tag{3.6b}$$

The stepsize τ_k determined by (3.6) basically ensures that ω_{k+1} reduces the objective $\psi(\omega)$ and, since Ω is convex, remains feasible. It is shown in Rustem (1992,1998) that (3.6) can always be fulfilled by the following algorithm.

The Algorithm

Step 0: Given $\omega_0 \in \Omega$, and small positive numbers ρ, γ, such that $\rho \in (0, 1)$, $\gamma \in (0, 1)$, the initial Hessian approximation, $\hat{H}_0$, set $k = 0$.

Step 1: Compute $\nabla \hat{J}_k$. Solve the quadratic subproblem (3.3) or (3.4) (choosing (3.3) or (3.4) defines a particular algorithm) to obtain d_k, λ_{k+1}, and the associated multiplier vectors. In (3.4), we also compute v_{k+1}.

Step 2: Test for optimality: If optimality is achieved, *stop*. Else, go to *Step 3*.

Step 3: Find the smallest nonnegative integer j_k such that $\tau_k = \gamma^{j_k}$, with ω_{k+1} given by (3.5), such that the inequality (3.6) is satisfied.

Step 4: Update $\hat{H}_k$ to compute $\hat{H}_{k+1}$, set $k = k + 1$ and go to *Step 1*.

$\hat{H}_k$ is evaluated using Powell's (1978) modification to the BFGS quasi-Newton formula (Broyden, 1969, 1970; Fletcher, 1970; Goldfarb, 1970; Shanno, 1970).

In Rustem (1992) and Rustem and Nguyen (1998), it is shown that d_k is a descent direction; the algorithm converges, at a Q-superlinear rate, to a solution of the min–max problem and the stepsize τ_k converges to unity.

4. Rival risk and return scenario models

In the original min–max framework of (2.1), rival forecast scenarios may be associated with common risk over all scenarios, or corresponding rival benchmark/risk scenarios, or there may exist rival risk scenarios, independent of rival returns. We consider the following five portfolio models (M–m: Min–max):

- Simple mean–variance: M^0 (Fixed Return; Fixed Risk)
- M–m rival return scenarios: M^1 (Rival Return Scenarios; Fixed Risk)
- M–m rival return-risk scenario *pairs*: M^2 (Rival Return-Risk Scenario *Pairs*)
- M–m rival return and rival risk scenarios: M^3 (Rival Return Scenarios; Rival Risk Scenarios)
- M–m rival benchmark/risk scenarios: M^4 (Fixed Return; Rival Benchmark/Risk Scenarios)

M^1 (Rival Return Scenarios; Fixed Risk). The simplest multiple scenario model is the case with rival return forecasts. To illustrate this, consider the ith scenario given by $\mathscr{E}(r^i) \in \mathbb{R}^n$ and

$$J^i_\alpha(\omega) = -\langle \mathscr{E}(r^i), \omega \rangle + \alpha \langle \omega - \bar{\omega}, \Lambda(\omega - \bar{\omega}) \rangle \tag{4.1a}$$

with

$$r^i = \mathscr{E}(r^i) + \varepsilon \tag{4.1b}$$

and, as in (1.2), $\bar{\omega}$ is the vector of benchmark weights. Hence, in (4.1) the *risk model is assumed to be common* across all the scenarios. Let $R \in \mathbb{R}^{n \times m^{\text{sce}}}$ be the matrix

$$R = [\mathscr{E}(r^1) \vdots \cdots \vdots \mathscr{E}(r^i) \vdots \cdots \vdots \mathscr{E}(r^{m^{\text{sce}}})]. \tag{4.2a}$$

For (4.1), the min–max problem (2.4) can be expressed as

$$\min_{\omega, v \in \mathbb{R}^{n+1}} \{ -v + \alpha \langle \omega - \bar{\omega}, \Lambda(\omega - \bar{\omega}) \rangle | \omega \in \Omega; \quad R^{\mathrm{T}} \omega \geq 1^\lambda v \} \tag{4.2b}$$

which is a *quadratic programming problem*. Algorithms for solving (4.2b) are discussed, for example, in Gill et al. (1981).

Min–max is a robust strategy with respect to the specified scenarios. It cannot be expected to be robust to general unspecified scenarios. Yet, it does provide some cover: robustness to certain scenarios, not specified in the problem formulation, can be assured. Let $\mathscr{E}(\bar{r})$ be a *return* vector *not taken into consideration* in the scenario set $\mathscr{E}(r^i)$, $i = 1, \ldots, m^{\text{sce}}$, in (4.2).

The extent of robustness cover provided by the min–max strategy for scenarios, $\mathscr{E}(\bar{r})$, not included in min–max formulation is summarized in Lemma 3.

Lemma 3. Let

(i) ω_*, v_* *solve* (4.2); *and,*
(ii) $E(\bar{r}) = R\Psi$ *with* $\Psi \in \mathbb{R}_+^{m^{sce}}$.
Then, we have $\langle \mathscr{E}(\bar{r}), \omega_* \rangle \geq v_*$.

Proof. Premultiplying $R^T\omega_* \geq 1^\lambda v_*$ by Ψ yields the desired result. □

The robust character of min–max, in Lemma 2, is in the noninferiority of the strategy to any scenario other than those actively contributing to the min–max decision, with $\lambda_*^i \in (0, 1)$. Lemma 3 establishes the noninferiority of the strategy for all possible forecasts $\mathscr{E}(\bar{r})$ that can be expressed as a convex combination of the specified scenarios. Equivalently, the min–max strategy, ω_*, v_* would be unchanged if $\mathscr{E}(\bar{r})$ was added to problem (2.4) or (2.5) as an additional, $(m^{sce} + 1)$st, scenario.

M^2 (*Rival Return–Risk Scenario pairs*). The second min–max portfolio model arises *when each rival return forecast has an associated risk*. Thus, each scenario consists of a return–risk pair. Consider, therefore,

$$J_\alpha^i(\omega) = -\langle \mathscr{E}(r^i), \omega \rangle + \alpha \langle \omega - \bar{\omega}, \Lambda^i(\omega - \bar{\omega}) \rangle \tag{4.3}$$

where, instead of (4.1b), we have

$$r^i = \mathscr{E}(r^i) + \varepsilon^i \tag{4.4}$$

and $\varepsilon^i \sim \mathscr{N}(0, \Lambda^i)$, $\Lambda^i \in \mathbb{R}^{n \times n}$. The min–max problem is thus given by (2.1) with the objective (4.3)

$$\min_{\omega} \max_{i} \{ -\langle \mathscr{E}(r^i), \omega \rangle + \alpha \langle \omega - \bar{\omega}, \Lambda^i(\omega - \bar{\omega}) \rangle | i = 1, \dots, m^{\text{sce}}; \quad \omega \in \Omega \}, \tag{4.5}$$

or by the nonlinear programming problem (2.4)

$$\min_{\omega, v \in \mathbb{R}^{n+1}} \{ v | \{ -\langle \mathscr{E}(r^i), \omega \rangle + \alpha \langle \omega - \bar{\omega}, \Lambda^i(\omega - \bar{\omega}) \rangle \} \leq v, \quad i = 1, \dots, m^{\text{sce}}; \quad \omega \in \Omega \}. \tag{4.6}$$

M^3 *(Rival Return Scenarios; Rival Risk Scenarios)*. The third possibility arises when *rival forecast return scenarios and rival forecast risk scenarios exist relatively independently* and it is desired to compute the best investment strategy that takes account of the worst-case of the return as well as the risk scenarios. Hence, consider the problem

$$\min_{\omega} \max_{i,j} \{ -\langle \mathscr{E}(r^i), \omega \rangle + \alpha \langle \omega - \bar{\omega}, \Lambda^j(\omega - \bar{\omega}) \rangle | \omega \in \Omega; \quad i = 1, \dots, m^{\text{sce}^r}; \quad j = 1, \dots, m^{\text{sce}^\Lambda} \} \tag{4.7}$$

where m^{sce^r} and m^{sce^Λ} are the numbers of return and risk scenarios respectively. The equivalent nonlinear programming problem to (4.7) is given by

$$\min_{\omega, v, \phi \in \mathbb{R}^{n+2}} \{ -v + \alpha\phi | R^{\mathrm{T}}\omega \geq 1^{\lambda} v; \langle \omega - \bar{\omega}, \Lambda^j(\omega - \bar{\omega}) \rangle \leq \phi;$$

$$j = 1, \dots, m^{\mathrm{sce}^\Lambda}; \quad \omega \in \Omega \}, \tag{4.8}$$

where $\phi \in \mathbb{R}^1$, 1^λ is the m^{sce^r}-dimensional vector whose elements are all unity and $R \in \mathbb{R}^{n \times m^{\mathrm{sce}^r}}$ is a matrix similar to (4.2a). In (4.7)–(4.8), for given α, the *worst-case return* $\langle \mathscr{E}(r^i), \omega \rangle$, given by v, and the *worst-case risk* $\langle \omega - \bar{\omega}, \Lambda^j(\omega - \bar{\omega}) \rangle$, given by ϕ, are identified for some i, j.

The results below establish that the min–max problems (4.1)–(4.2), (4.5)–(4.6) and (4.7)–(4.8) are continuous in terms of the maximizers v and ϕ as α changes and determine the Markowitz efficiency of the v–ϕ frontier for (4.1)–(4.2) and (4.7).

Lemma 4. Let $[\omega_*(\alpha), v_*(\alpha)]$ *denote the solutions of* (4.1)–(4.2) *and* (4.5)–(4.6). *Let* $[\omega_*(\alpha), v_*(\alpha), \phi_*(\alpha)]$ *denote the solution of* (4.8). *Let the Jacobian matrices associated with the first-order optimality conditions of* (4.1)–(4.2), (4.5)–(4.6) *and* (4.8) *be nonsingular. Then* $[\omega_*(\alpha), v_*(\alpha)]$ *and* $[\omega_*(\alpha), v_*(\alpha), \phi_*(\alpha)]$ *are continuous in* α.

Proof. The proof follows from Fiacco (1976, Corollary 2.1).

Lemma 5. The frontier of $v_*(\alpha)$ *and* $\langle \omega_*(\alpha) - \bar{\omega}, \Lambda(\omega_*(\alpha) - \bar{\omega}) \rangle$ *determined by* (4.1)–(4.2) *and the frontier of* $v_*(\alpha)$ *and* ϕ_* *determined by* (4.8) *are Pareto (Markowitz), efficient.*

Proof. The proof follows from the convexity of problems (4.1)–(4.2) and (4.8). □

M^4 *(Fixed Return; Rival Benchmark/Risk Scenarios).* The fourth possibility arises from rival benchmark weights and rival risk scenarios. The investor may wish to consider a portfolio (i.e. ω) *that attempts to track more than one set of benchmark weights* (i.e. ω). In this case, min–max would try to compute the best set of weights corresponding to the worst departure from the given set of benchmarks. A similar problem is due to different risk scenarios, in the form of *rival variance* estimates. Consider, therefore, the problem

$$\min_{\omega} \max_{i,j} \{ -\langle \mathscr{E}(r), \omega \rangle + \alpha \langle \omega - \bar{\omega}^j, \Lambda^i(\omega - \bar{\omega}^j) \rangle | j = 1, \dots, m^{\mathrm{sec}^j},$$

$$i = 1, \dots, m^{\mathrm{sec}^i}; \omega \in \Omega \}, \tag{4.9}$$

where $\bar{\omega}^j$ is the jth rival benchmark and Λ^i is the ith rival variance.

Clearly, there are further possible combinations of the above portfolio models. Their efficiency can be similarly evaluated. For example, in the case of (4.9), interpreted as *min–max Model*[3] with $i = 1$ in (4.7), the efficiency discussed in Lemma 5 applies.

The only min–max portfolio case for which Markowitz efficiency cannot be established, as α varies, is *min–max portfolio Model*[2] given by (4.5)–(4.6). In this case, the noninferiority property of min–max in Lemma 2 ensures that

$$\begin{aligned} &-\langle \mathscr{E}(r^{\ell}), \omega_*(\alpha)\rangle + \alpha\langle \omega_*(\alpha) - \bar{\omega}, \Lambda^{\ell}(\omega_*(\alpha) - \bar{\omega})\rangle \\ &\quad \leq -\langle \mathscr{E}(r^{j}), \omega_*(\alpha)\rangle + \alpha\langle \omega_*(\alpha) - \bar{\omega}, \Lambda^{j}(\omega_*(\alpha) - \bar{\omega})\rangle \end{aligned} \tag{4.10}$$

$\forall \lambda_*^{\ell} = 0$, $\lambda_*^{j} \in (0, 1)$. In other words, min–max assures a basic optimal performance, defined by the utility functions corresponding to the worst-case (maximizing) scenario, indicated by index j. This performance is maintained, or improved, if any other scenario, not active in the min–max strategy and indicated by index ℓ, happens to represent the future. However, the relation between $\langle \mathscr{E}(r^{j}), \omega_*(\alpha)\rangle$ and $\langle \omega_*(\alpha) - \bar{\omega}, \Lambda^{j}(\omega_*(\alpha) - \bar{\omega})\rangle$, as α varies, cannot be assured to be Markowitz efficient. One important issue in this respect is the fact that if there is more than one i, j that corresponds to the maximum, then λ_*^{i}, $\lambda_*^{j} \in (0, 1)$ and

$$\begin{aligned} &-\langle \mathscr{E}(r^{i}), \omega_*(\alpha)\rangle + \alpha\langle \omega_*(\alpha) - \bar{\omega}, \Lambda^{i}(\omega_*(\alpha) - \bar{\omega})\rangle \\ &\quad = -\langle \mathscr{E}(r^{j}), \omega_*(\alpha)\rangle + \alpha\langle \omega_*(\alpha) - \bar{\omega}, \Lambda^{j}(\omega_*(\alpha) - \bar{\omega}\rangle. \end{aligned} \tag{4.11}$$

The risk–return frontiers $\{\langle \mathscr{E}(r^{j}), \omega_*(\alpha)\rangle; \langle \omega_*(\alpha) - \bar{\omega}, \Lambda^{j}(\omega_*(\alpha) - \bar{\omega})\rangle\}$ and $\{\langle \mathscr{E}(r^{i}), \omega_*(\alpha)\rangle; \langle \omega_*(\alpha) - \bar{\omega}, \Lambda^{i}(\omega_*(\alpha) - \bar{\omega})\rangle\}$ would be different, in general. The frontiers corresponding to all i, j have the same utility, as long as (4.11) is satisfied. Nevertheless, these frontiers are not Markowitz efficient. This is essentially the price paid to achieve the equality in (4.11): if there is a single maximizer such that there is no equality among utilities and one j dominates all other scenarios, then it can be verified that the $\langle \mathscr{E}(r^{j}), \omega_*(\alpha)\rangle$ and $\langle \omega_*(\alpha) - \bar{\omega}, \Lambda^{j}(\omega_*(\alpha) - \bar{\omega})\rangle$ frontier is also Markowitz efficient. In the case when (4.11) is achieved between at least two risk–return scenarios, the loss of efficiency needs to be evaluated in view of the robustness it brings for given α. The efficient investment strategy for any particular scenario, for that α, would lead to a deteriorated performance that is at best the same as min–max, or considerably worse, if any of the other scenarios were realized.

The frontiers corresponding to equality (4.11) are plotted for comparison with the corresponding efficient strategy for each scenario. Then, we measure the deterioration in performance if another scenario is realised (e.g. Fig. 8 below).

5. Numerical experiments

As an application of the ideas developed in this paper, we consider an international bond portfolio problem using three classes of assets in sixteen currencies. The benchmark weights $\bar{\omega}$ are chosen to track the time series of JP Morgan world government bond index in return terms. The assets are six month money market instruments, five year and ten year (benchmark) bonds in USA (T), Canada, Australia, Japan (JGB), Germany (BUND), Holland (DSL), France (BTAN), Belgium (OLO), Denmark (DGB), UK (Gilt), ECU, Italy (BTP), Spain (Bono), Sweden (SGB), Switzerland, New Zealand.

5.1. Notation

In the figures below, '*absolute return*' indicates $\langle \mathscr{E}(r), \omega_* \rangle$ and is plotted as % per annum, as $\mathscr{E}(r)$ is represented in % per annum. Also, '*relative risk*'/'*active risk*' indicates

$$\sqrt{\langle \omega_* - \bar{\omega}, \Lambda(\omega_* - \bar{\omega}) \rangle}$$

and, as the risk of the return, it is expressed in % per annum. The following notation is used throughout:

Model^0: M^0 *(Return Risk)*. Model^0 is the simple mean–variance optimization (1.1)–(1.2), given a return and a risk estimate. Often, this provides the basis for comparison with other evaluations and strategies. M^0 (Return Risk) indicates the particular Return and Risk used in the optimization. It corresponds to a strategy (i.e. a set of weights) and a performance represented by an efficient risk-return frontier.

Model^1: M^1 ([*Rival-Return*]^1 … [*Rival-Return*]^i …; *Risk*). Model^1 is the min–max evaluation with rival return estimates and a single risk. Consequently, M^1 ([Rival–Return]^i; Risk) indicates the particular return scenarios and risk used to compute the min–max strategy. The result of M^1 is a strategy (i.e. a set of weights) and a guaranteed lower bound on the efficient performance (i.e. risk-return frontier) of the strategy. This lower bound corresponds to the worst-case and performance may only improve if the worst-case scenario is not realised.

Model^2: M^2 ([*Rival-Return Rival-Risk*]^1 … [*Rival-Return Rival-Risk*]^i …). Given rival risk and return pairs, this computes the min–max strategy for all. It corresponds to a strategy (i.e. a set of weights). However, there is no unique performance measure. Each risk and return pair yields a corresponding mean–variance performance. Neither performance is guaranteed to be efficient. The performance of the strategy is evaluated given each scenario.

Model^3: M^3 ([*Rival-Return*]^i [*Rival-Risk*]^j, $i, j = 1, 2, \ldots$). With the rival return and rival risks considered as individual scenarios, the min–max strategy is evaluated for the worst-case i and j. The strategy also corresponds to a worst-case performance efficient frontier. As in M^1, M^3 provides a guaranteed lower bound on performance which may only improve (or remain the same) if any scenario, other than the worst-case, is realised.

Model[4]: M^4 (*Return*; [*Rival-Risk*]^1 … [*Rival-Risk*]^i …). For a given return and rival risk scenarios, the min–max strategy yields a similar guaranteed lower bound efficient frontier to M^1 and M^3.

Evaluation: E (*Return*^{i} *Risk*^{j}; M^m) $m = 0, 1, 2, 3, 4$. Each Model M^m discussed above corresponds to an investment or portfolio strategy based on certain scenario(s). We are sometimes interested in evaluating the effect of implementing this strategy assuming a scenario is realised. Hence, E (Return^i Risk^j; M^m) is the performance evaluation of the strategy based on Model M^m if the scenarios Return^i, Risk^j are realised.

Example: E (*Return*^{i} *Risk*^{j}; M^0 (*Return*^{k} *Risk*^{l})). This is the performance evaluation of the optimal mean–variance strategy M^0(Return^k Risk^l) using another scenario: Return^i Risk^j. The result is a risk-return frontier indicating the consequence of implementing the optimal strategy based on (k l) if the rival scenario (i j) is realised. Thus, $E(DF;M^0(PF))$ is the frontier arising from the evaluation of the mean–variance strategy based on return scenario P, risk scenario F when return scenario D and risk scenario F is realised (see Fig. 3).

5.2. *Model*[1]: M^1 (*Core Doom Prosperity*; *Forecast Risk*)

Using the interest rate and exchange rate forecasts for each country, adopting US$ as the base currency, three different return scenarios are obtained. The first of these is the '*core*' *(C)* scenario, or the basic expected state for the overall system. The second is the '*prosperity*' *(P)* scenario, representing a better than average performance of the underlying economies. The third is the '*doom*' (D) scenario, representing the pessimistic point of view. The risk is identified with the past covariance of the '*forecast*' *errors (F)*. This contrasts with the '*historical*' *covariance (H)* of the actual instruments used in later models. The model is thus denoted by $M^1(PCD; F)$.

Remark 1. Each risk–return frontier starts at the lower end, near the origin, with the portfolio return that can be achieved at lowest risk (i.e. as $\alpha \to \infty$) and ends with the highest return with the highest risk (i.e. with $\alpha = 0$). Thus, the two ends of each frontier may be different if, for the given scenario(s) and optimal strategy, the portfolio return values corresponding to these two end points are different.

Fig. 1 illustrates the basic guaranteed return represented by 'min–max lower bound' $M^1(PCD; F)$. This is an efficient frontier. Near the risk-averse end of the frontier, it is based on the 'doom' scenario. Subsequently, it becomes jointly based on both the 'core' and 'doom' scenarios. When the 'doom' scenario is the sole determinant of the strategy, we observe that if any of the other two

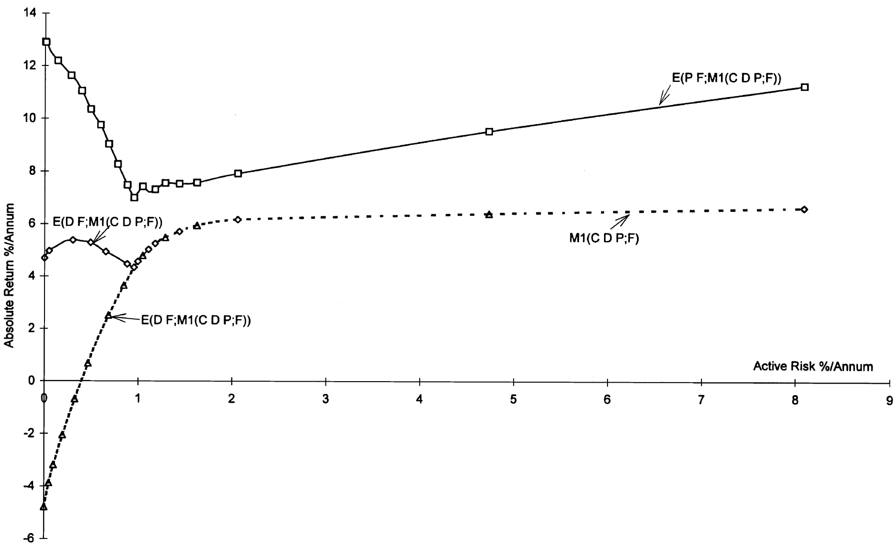

Fig. 1. Model 1, the min–max strategy and its evaluation with all scenarios.

scenarios are realised, the min–max strategy results in an improved performance, represented by the graphs lying above the min–max lower bound. When 'core' and 'doom' scenarios determine the min–max strategy, the basic guaranteed min–max performance is the same for either of these, and improves if 'prosperity' is realised. For a given level of risk, represented by the horizontal axis, the performance of the min–max strategy has a direct interpretation in terms of the three multipliers associated with the scenarios in Fig. 2 (see Lemma 2). The initial part, when 'doom' determines the strategy, the multiplier

$$\lambda(DF; M^1(CPD; F))$$

corresponding to 'doom' is unity while the other two multipliers are zero. At this stage, Parts (c) and (d) of Lemma 2 also assert that the portfolio returns corresponding to the zero multipliers would be greater than 'doom'. When 'doom' and 'core' jointly determine the strategy, the corresponding multipliers $\lambda(CF; M^1)$, $\lambda(DF; M^1)$ are nonzero, sum to unity and the returns are the same, while 'prosperity' multiplier $\lambda(PF; M^1)$ has null value and the return is higher if this scenario is realised (see Parts (a) and (b) of Lemma 2).

Figs. 3–5 illustrate the efficient mean–variance frontiers for each individual scenario (i.e. $M^0(PF)$, $M^0(DF)$, $M^0(CF)$) and evaluate the performance of these portfolio strategies if any of the other scenarios are realized. Any loss thereby incurred is contrasted with the guaranteed lower bound of the min–max strategy. In Fig. 3, the optimal 'prosperity' strategy deteriorates substantially if any of

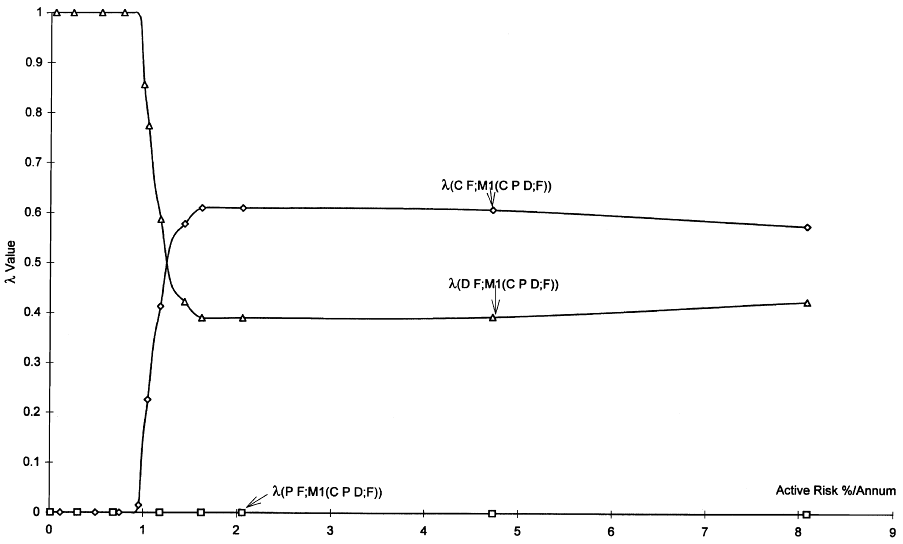

Fig. 2. Model 1 multiplier lambdas vs risk in the min–max strategy.

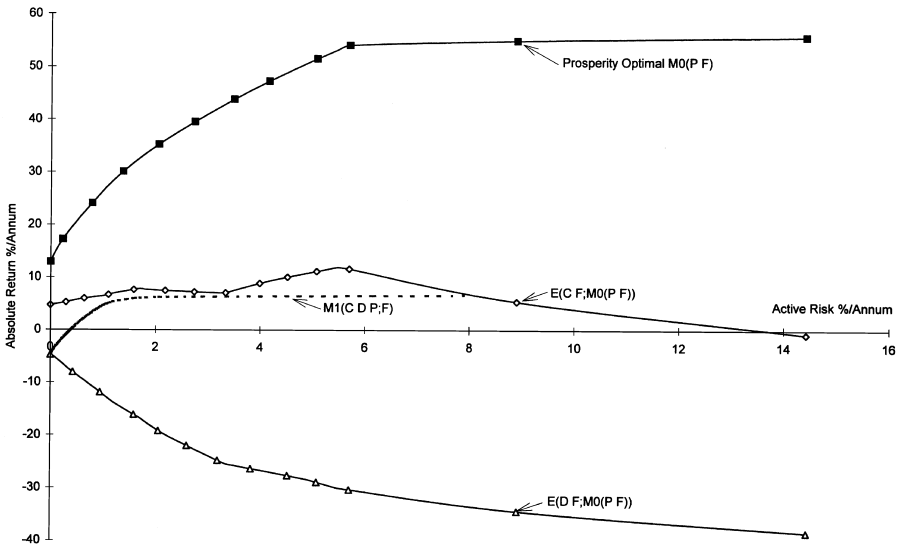

Fig. 3. Prosperity optimal strategy, cross evaluations comparison with Model 1.

the other two scenarios are realised. The 'doom' scenario E(DF; $M^0(PF)$) is especially disastrous and the 'core' scenario E(CF; $M^0(PF)$) is close to the min–max guaranteed lower bound. The 'doom' scenario starts with min–max as the latter is initially based on 'doom'. We note that, as explained in Remark 1,

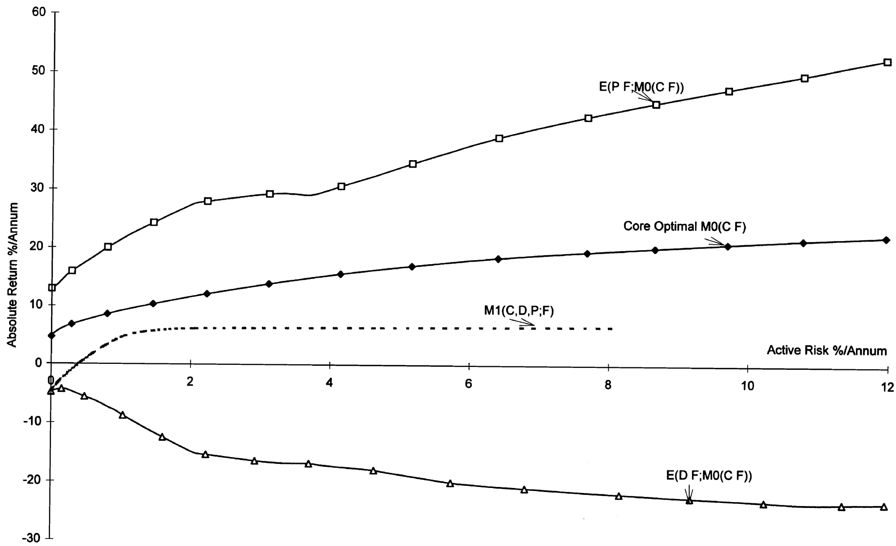

Fig. 4. Core optimal strategy, cross evaluations comparison with Model 1.

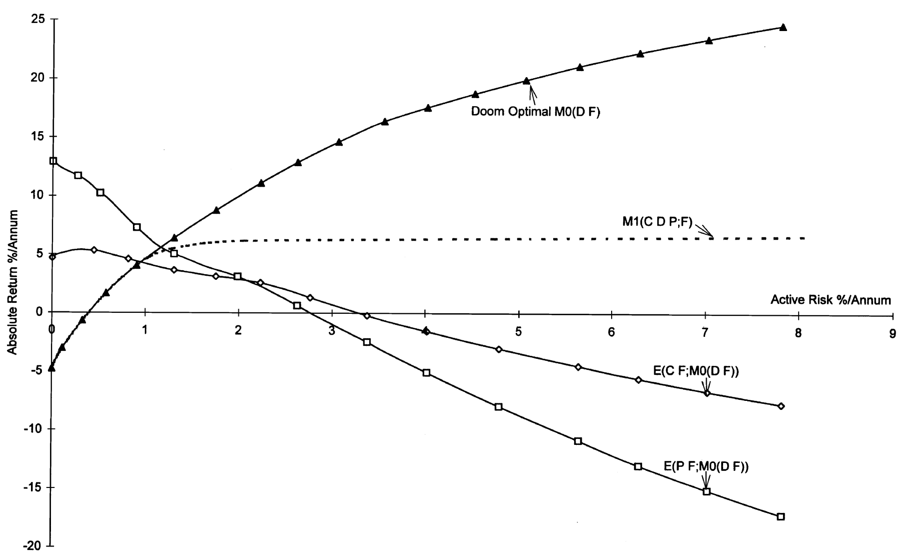

Fig. 5. Doom optimal strategy, cross evaluations comparison with Model 1.

the min–max portfolio terminates at a lower risk level as that corresponds to the highest return achieved under min–max. Indeed, it can be verified in Figs. 3–5 that, when evaluated against a different strategy, both ‘core’ and ‘doom’ scenarios, which determine min–max, lead to lower performance than min–max after this point.

In the absence of any special diversification constraints in Ω, the risky portfolio in the ordinary Markowitz mean–variance analysis, given by (1.1) with $\alpha = 0$, corresponds to a trivial linear programming problem solved by investing the full wealth in the asset with highest return. A special feature of min–max is that even in this risky situation, the investment is generally diversified if more than one scenario determines the strategy. The reason for this can be verified in (2.4) when more than one element of $\hat{J}(\omega) \leq 1^{\lambda} v$ is satisfied as equality at the solution. This feature was also observed in the numerical experiments.

In Fig. 4, the optimal 'core' strategy provides a performance close to min–max. This improves if the 'prosperity' scenario is realized but deteriorates well below 'min–max' when 'doom' is realised.

In Fig. 5, the optimal 'doom' strategy is evaluated. In this figure, we can also identify a *potentially suboptimal worst-case*: given the single-scenario optimal strategies in Figs. 3–5, we evaluate the deterioration in performance if any of the other two scenarios are realised. *The single-scenario optimal portfolio that leads to the least deterioration of performance is a potentially suboptimal worst case*. In this experiment, optimal 'doom' corresponds to this worst case. Initially, at the lower risk end, this happens to coincide with min–max and it is clearly efficient. After the min–max and doom strategies separate, this worst case continues as the optimal 'doom' strategy, with its performance evaluated relative to 'core' and then 'prosperity' as the scenarios with least deterioration. It can be observed that the suboptimal worst case becomes inefficient after it departs from the 'doom' optimal and min–max strategies as it corresponds to lower return for higher risk. Consequently, the suboptimal worst case would clearly not be implemented after this deviation. This is in contrast to the min–max strategy that is well defined, efficient and continues to provide a guaranteed lower bound after the deviation.

To illustrate the robust nature of min–max, Fig. 6 plots the mean–variance utility function values (i.e. the combination of mean and variance for a given α). Each point on these curves corresponds to the solution of (2.6) for a given λ. Thus, the individual utility functions corresponding to each scenario are plotted. In addition, for the given value of λ, the combination of the utilities are plotted. It can be verified that for $\lambda = 0$, the strategy is based on $M^0(DF)$ and for $\lambda = 1$, it is based on $M^0(CF)$. As λ is increased from 0 to 1, the combination of the utility functions

$$\lambda U(CF; M^1(CD; F)) + (1 - \lambda) U(DF; M^1(CD; F))$$

passes through a maximum with respect to λ where

$$U(CF; M^1(CD; F)) = U(DF; M^1(CD; F)).$$

This point corresponds to the min–max solution (see Lemma 2).

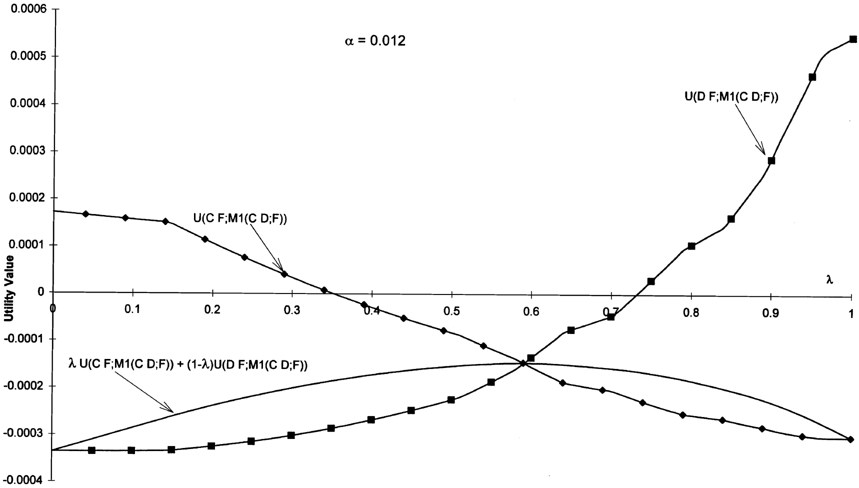

Fig. 6. The robust nature of min-max.

5.3. *Model*[2]: M^2 (*C Forecast-Risk, D Historical-Risk*)

Rival scenarios considered are the 'core' return with the covariance of 'forecast' errors as the risk and 'doom' return with the 'historical' covariance of the returns. We denote this model by $M^2(CF, DH)$. The min–max strategy is based on the worst-case scenario for a given α. Hence, there are two performance frontiers corresponding to the two scenarios. These are plotted in Fig. 7 as $\mathrm{E}(CF;\ M^2(CF, DH))$ and $\mathrm{E}(DH;\ M^2(CF,\ DH))$. Both evaluations indicate the performance of the strategy for the corresponding scenario. Hence, the investor would need to consider and decide on the basis of both frontiers. The plots also confirm the earlier observation that this strategy would not necessarily lead to an efficient frontier. However, min–max assures that the basic utility of the investor, measured by the combination of risk and return, for any given value of α, may only improve if the worst-case scenario is not realized (see (4.10)). For comparison, the M^1 strategies for rival return and single risk models used in this case are also plotted. The basic guaranteed return insurance provided by this strategy is illustrated in Fig. 8, comparing the efficient mean–variance frontiers $M^0(D;\ H)$, $M(C;\ F)$ and their cross evaluations $\mathrm{E}(CF; M^0(D;\ H))$ and $\mathrm{E}(DH;\ M^0(C;\ F))$. The deterioration observed in the latter two can be avoided by considering the robust strategy due to M^2.

5.4. *Model*[3]: M^3 (*CF-Risk, DH-Risk, CH-Risk, DF-Risk*)

In this case, two types of rival scenarios are considered: return scenarios and risk scenarios. Specifically, we consider the model $M^3(CF,\ DH,\ CH,\ DF)$. The

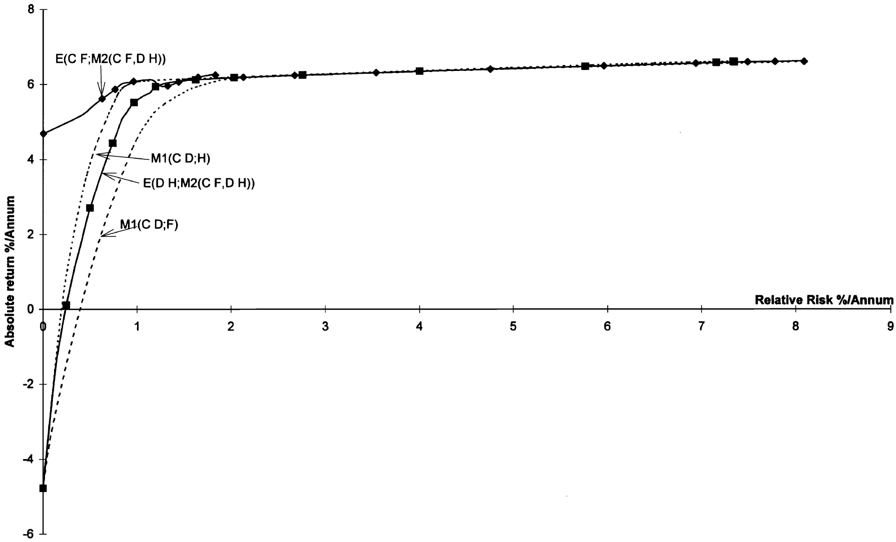

Fig. 7. Model 2 evaluations.

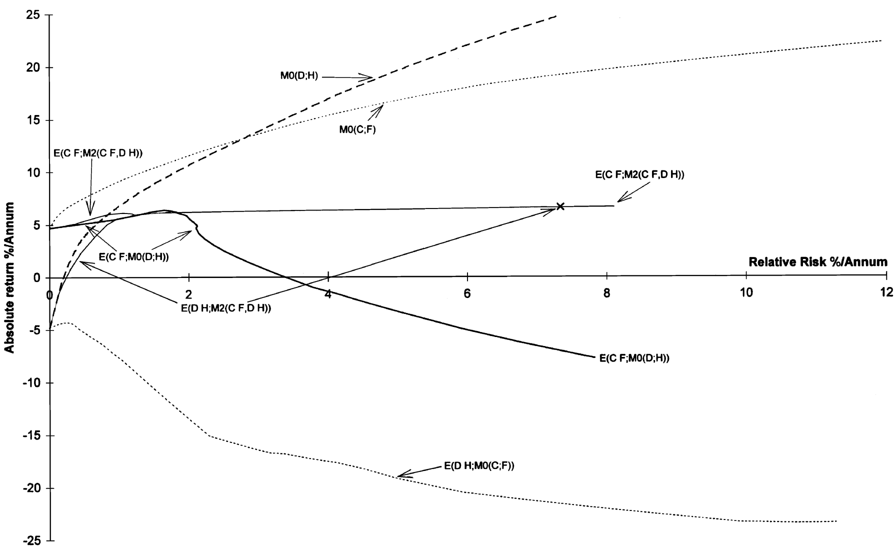

Fig. 8. Model 2 and comparison with M^0 strategies.

result is a strategy which yields an efficient risk-return frontier representing a basic guaranteed performance given the scenarios. The result is plotted in Fig. 9 with $M^1(CD; F)$, $M^1(CD; H)$ for comparison. The M^1 strategies for single risk are rather close to M^3, with $M^1(CD; F)$ and M^3 coinciding. Clearly, the

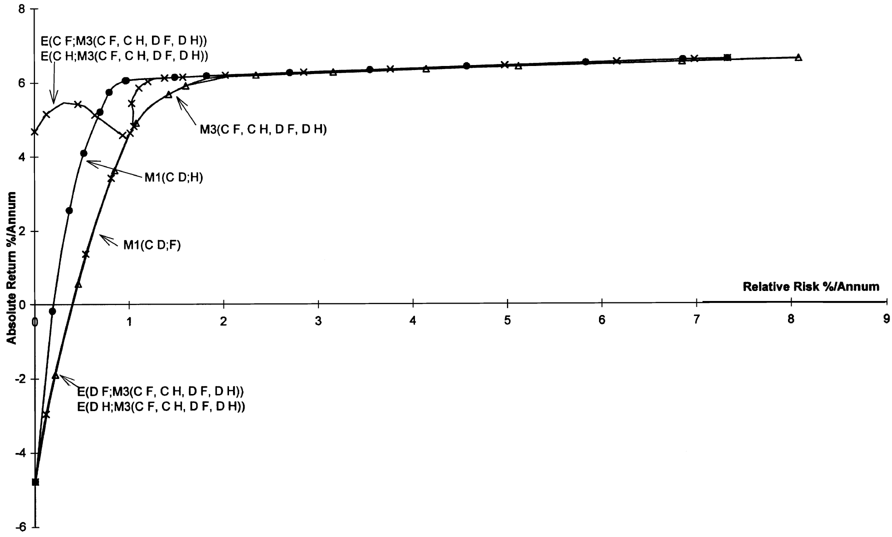

Fig. 9. Model 3 evaluations and the non-inferiority of min–max in view of the other scenarios.

forecast risk is dominant over the risk scenarios and this is the sort of indication that might be used to alert a forecaster. This figure also indicates evaluations E(CF; M^3), E(CH; M^3), E(DF; M^3), E(DH; M^3). This illustrates that the performance of the strategy can only improve from the basic guaranteed one provided by M^3, if any scenario other than the worst case is realised.

Fig. 10 indicates the differences in the strategies for a given α. For M^1 with different risk estimates and M^3, the portfolio allocation varies considerably.

5.5. Model^4: M^4 (C; F-Risk H-Risk) and M^4 (D; F-Risk H-Risk)

We consider two single return and multiple risk models: $M^4(C; FH)$ and $M^4(D; FH)$. As in M^1 and M^3, the result is a strategy which gives rise to an efficient risk-return frontier for each M^4. Each frontier represents a guaranteed performance in view of the scenarios considered. In both cases, the basic guaranteed min–max performance is based on $M^0(CF)$, $M^0(CH)$ and $M^0(DF)$, $M^0(DH)$ and these are given in Fig. 11. The min–max strategy corresponding to $M^4(C; FH)$ and optimal strategies $M^0(CF)$, $M^0(CH)$ are given in Fig. 12: M^4 corresponds to the guaranteed performance which can only improve if any other scenario, other than the worst case is realised. The corresponding case for $M^4(D; FH)$ is given in Fig. 13, with $M^0(DF)$, $M^0(DH)$. This case is of interest as the guaranteed mean–variance performance seems to be obtained using a combination of $M^0(DF)$, $M^0(DH)$. Actually, although the min–max strategy is able

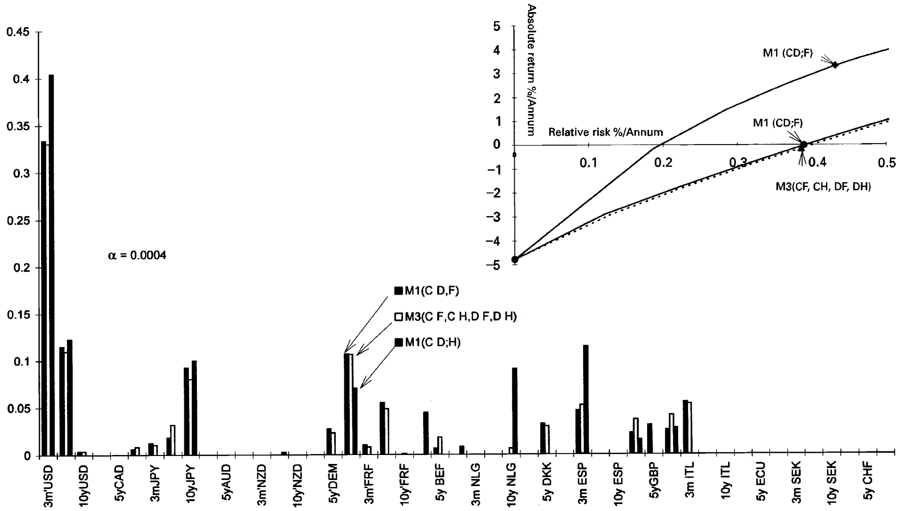

Fig. 10. Differences in minimax strategies.

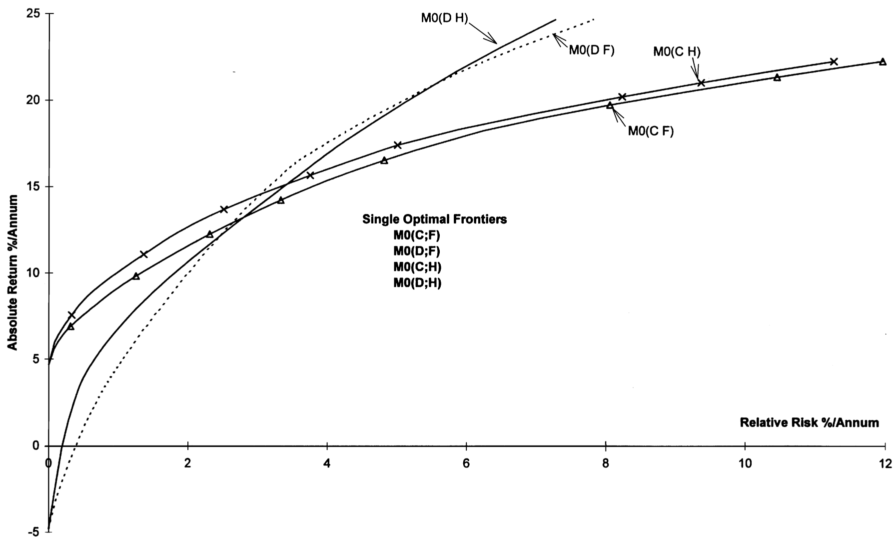

Fig. 11. Model 0 performance for four scenarios used in Model 4.

to ensure this performance, it has the noninferiority property: whichever scenario is realised, the performance will not deteriorate.

The cross evaluations of $M^0(DF)$, $M^0(DH)$, given by $\mathrm{E}(DF;M^0(DH))$, $\mathrm{E}(DH;M^0(DF))$, indicate that performance deteriorates considerably for the

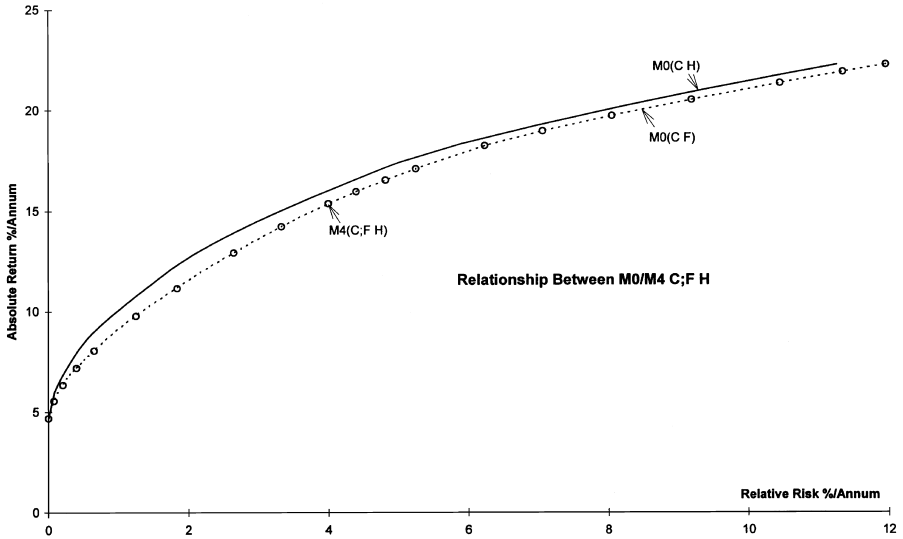

Fig. 12. Model 4 with 'core' return forecast as a guaranteed basic performance for (CF) (CH).

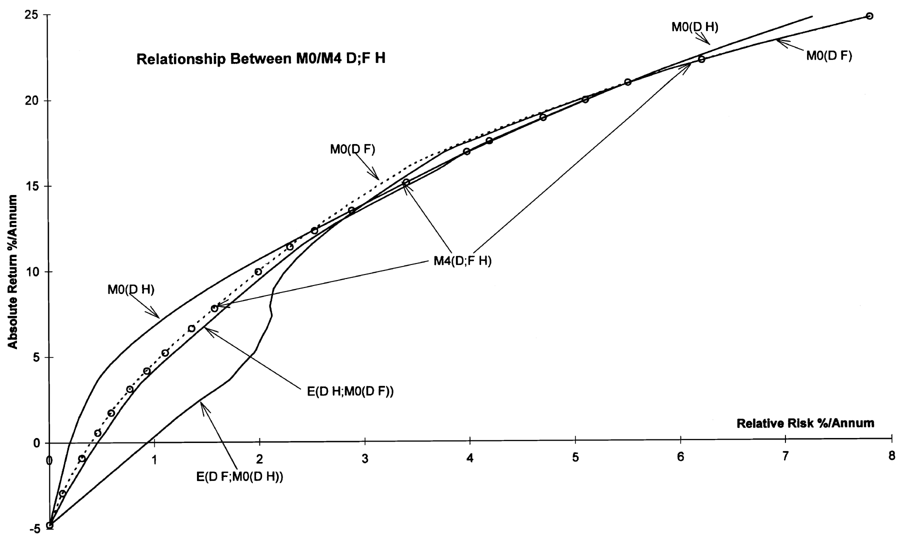

Fig. 13. Model 4 with 'Doom' return forecast as a guaranteed basic performance for (DF) (DH).

optimal strategies based on a single scenario. The comparison of both cases with $M^1(CD; F)$ and $M^1(CD; H)$ is given in Fig. 14. The worst-case analysis of return scenarios, given each risk estimate, seems to have led mostly to a more cautious strategy than the worst-case analysis of the risk scenarios, for each return

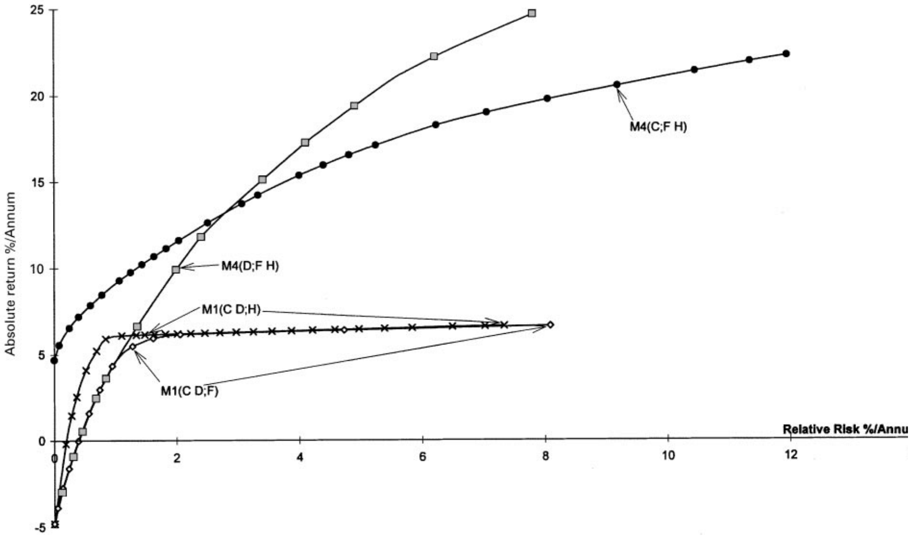

Fig. 14. Model 4 evaluation, comparison with Model 1.

estimate. Furthermore, for higher levels of risk, the relative performance of the two M^4 strategies seem to switch and $M^4(D; FH)$ overtakes $M^4(C; FH)$. The reason is that 'doom' seems to result in higher returns over 'core' at high levels of risk, as illustrated by the performance of the individual optimal strategies in Fig. 11.

The multipliers λ for $M^4(C; FH)$ and $M^4(D; FH)$ are plotted in Figs. 15 and 16. The switches in λ in this figure correspond to the switches in the dominant strategy determining the guaranteed performance, in Fig. 13. The corresponding numerical result for $M^4(C; FH)$ is omitted. In this case, the dominance of one strategy, in determining the guaranteed performance, is observed by the selection of $\lambda(CF, M^4(C; FH)) = 1$ in most cases. Finally, Fig. 17 illustrates the portfolio holdings of the min–max strategy $M^4(D; FH)$ and the optimal strategy $M^0(DF)$. While the two have a very similar risk-return performance, as illustrated in Fig. 13, the underlying portfolio holdings are slightly different. As mentioned above, the min–max strategy performance is guaranteed whereas the optimal strategy M^0 may deteriorate if the alternative scenario is realised.

6. Concluding remarks

The min–max strategy discussed in this paper is essentially an addition to the arsenal of computational tools available to those interested in optimal

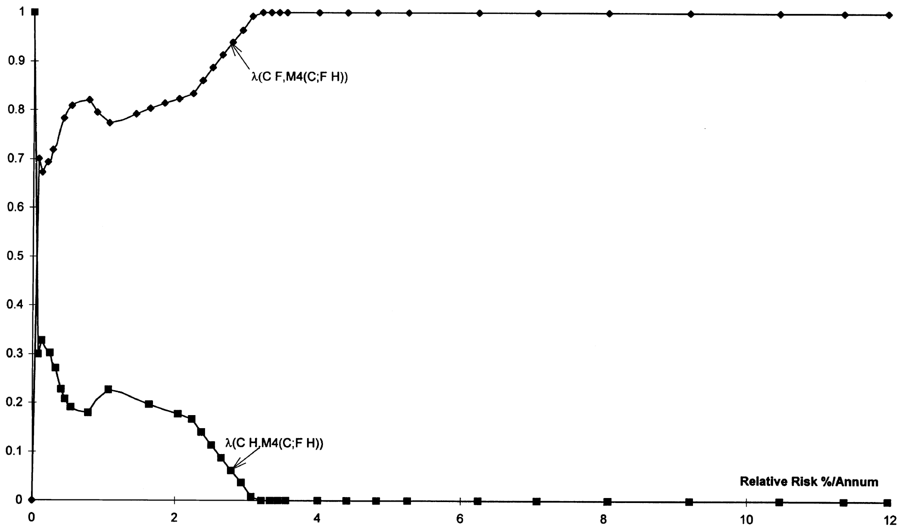

Fig. 15. Model 4 (C; FH) the multiplier lambdas vs risk.

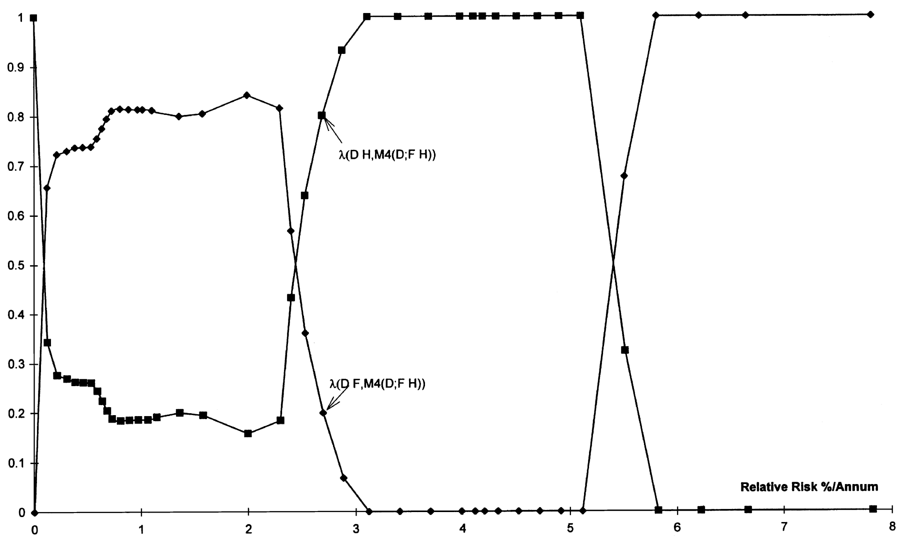

Fig. 16. Model 4 (D; FH) the multiplier lambdas vs risk.

portfolios. As in the case of the original Markowitz framework, the success of the present method depends on the precision of the data input to the problem. If the scenario choice is judicious, then the method is bound to be successful. If

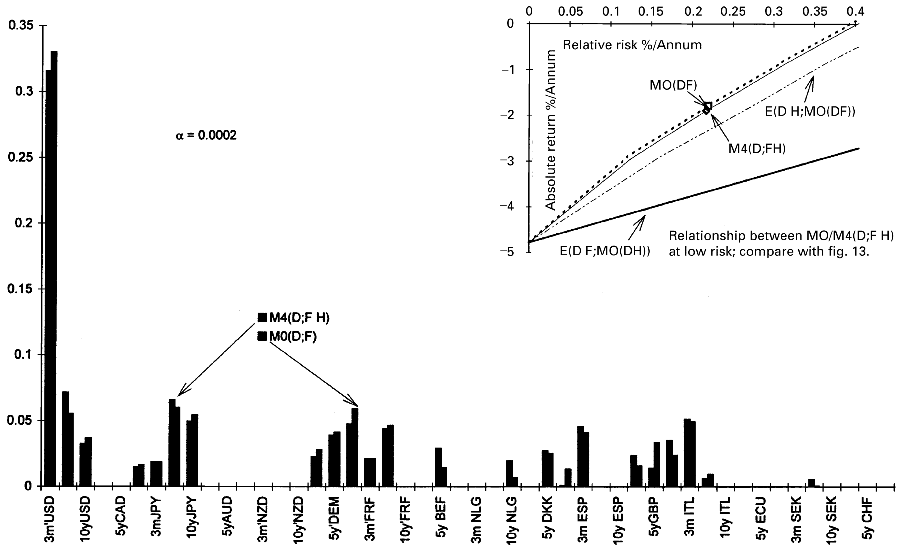

Fig. 17. Strategy comparison for $M^4(D; FH)$ and $M^0(D; F)$.

the method is used as a substitute to wise problem formulation, for example, by specifying as many scenarios as possible to cover all future contingencies, then the performance of the method will deteriorate. Either the worst case will be unnecessarily pessimistic or numerical problems will occur due to the linear dependence of the objective function gradient vectors corresponding to the (multiple) maximizing scenarios. There is, thus, no substitute for judicious investment analysis.

The present method enables the decision maker to specify the scenarios that seem to be likely and to compute the worst-case optimal investment portfolio. This portfolio ensures a basic return-risk performance, given the rival scenarios. According to Lemma 2, this performance is guaranteed to be maintained, or improved upon if any of the specified scenarios are realised.

Acknowledgements

The authors would like to acknowledge the comments of two referees, the guest editor, the active support of Rudi Bogni and Marc Hendriks, who recognised the potential of min–max as a decision tool to handle multiple forecast scenarios.

References

Becker, R.G., Dwolatzky, B., Karakitsos, E., Rustem, B., 1986. The simultaneous use of rival models in policy optimization. The Economic Journal 96, 425–448.

Broyden, C.G., 1969. A new method for solving nonlinear simultaneous equations. Computer Journal 12, 95–100.

Broyden, C.G., 1970. The convergence of a class of double-rank minimisation algorithms 2. The new algorithm. Journal of the Institute of Mathematics and its Applications 6, 222–231.

Charalambous, C., Conn, A.R., 1978. An efficient algorithm to solve the min–max problem directly. SIAM Journal of Numerical Analysis 15, 162–187.

Chow, G.C., 1979. Effective use of econometric models in macroeconomic policy formulation. In: Holly, S., Rustem, B., Zarrop, M. (Eds.), Optimal Control for Econometric Models. Macmillan, London.

Cohen, G., 1981. An algorithm for convex constrained minmax optimization based on duality. Applied Mathematics and Optimization 7, 347–372.

Coleman, T.F., 1978. A note on new algorithms for constrained minimax optimization. Mathematical Programming 15, 239–242.

Conn, A.R., 1979. An efficient second order method to solve the constrained minmax problem. Department of Combinatorics and Optimization, University of Waterloo, Report January.

Demyanov, V.F., Maloemzov, V.N., 1974. Introduction to Minmax. Wiley, New York.

Demyanov, V.F., Pevnyi, A.B., 1972. Some estimates in minmax problems. Kibernetika 1, 107–112.

Dutta, S.R.K., Vidyasagar, M., 1977. New algorithms for constrained minmax optimization. Mathematical Programming 13, 140–155.

Fiacco, A.V., 1976. Sensitivity analysis for nonlinear programming using penalty methods. Mathematical Programming 10, 267–311.

Fletcher, R., 1970. A new approach to variable metric algorithms. Computer Journal 13, 317–322.

Fuhrer, J., Haltmaier, J., 1986. Minimum variance pooling of forecasts at different levels of aggregation. Special Studies Paper 208, Federal Reserve Board, Washington, D.C.

Gill, P.E., Murray, W., Wright, M.H., 1981. Practical Optimization. Academic Press, New York.

Goffin, J.L., Haurie, A., Vial, J.P., 1992. Decomposition and nondifferentiable optimization with the projective algorithm. Management Science 38, 284–302.

Goldfarb, D., 1970. A family of variable metric algorithms derived by variational means. Mathematics of Computation 24, 23–26.

Granger, C., Newbold, P., 1977. Forecasting Economic Time Series. Academic Press, New York.

Han, S-P, 1981. Variable metric methods for minimizing a class of nondifferentiable functions. Mathematical Programming 20, 1–13.

Howe, M., Rustem, B., Selby, M., 1994. Minimax hedging strategy. Computational Economics 7, 245–275.

Howe, M., Rustem, B., Selby, M., 1996. Multi-period minimax hedging strategies. European Journal of Operational Research 93, 185–204.

Howe, M., Rustem, B., 1997. A robust hedging algorithm. Journal of Economic Dynamics and Control 21, 1065–1092.

Kall, P., Wallace, S.W., 1994. Stochastic Programming. Wiley, New York.

Lawrence, M.J., Edmunson, R.H., O'Connor, M.J., 1986. The accuracy of combining judgemental and statistical forecasts. Management Science 32, 1521–1532.

Makridakis, S., Winkler, R., 1983. Averages of forecasts: some empirical results. Management Science 29, 987–996.

Medanic, J., Andjelic, M., 1971. On a class of differential games without saddle-point solutions. Journal of Optimization Theory and Applications 8, 413–430.

Medanic, J., Andjelic, M., 1972. Minmax solution of the multiple target problem. IEEE Transactions AC-17, 597–604.

Murray, W., Overton, M.L., 1980. A Projected Lagrangian Algorithm for Nonlinear Minmax Optimization. SIAM Journal of Scientific and Statistical Computation 1, 345–370.

Pardalos, P.M., Sandstrom, M., 1994. On the use of optimization models for portfolio selection: A review of some computational results. Computational Economics 7, 227–244.

Polak, E., Mayne, D.Q., Higgins, J.E., 1988. A superlinearly convergent min–max algorithm for min–max problems. Memorandum No: UCB/ERL M86/103, Berkeley California.

Powell, M.J.D., 1978. A fast algorithm for nonlinearly constrained optimization calculations. In: Watson, G.A. (Ed.), Numerical analysis proceedings, Dundee, 1977. Springer, Berlin.

Rustem, B., 1987. Methods for the simultaneous use of multiple models in optimal policy design. In: Carraro, C., Sartore, D. (Eds.), Developments in Control Theory for Economic Analysis. Maritnus Nijhoff Kluwer Publishers, Dordrecht.

Rustem, B., 1992. A constrained min–max algorithm for rival models of the same economic system. Mathematical Programming 53, 279–295.

Rustem, B., 1998. Algorithms for Nonlinear Programming and Multiple Objective Decisions. Wiley, Chichester.

Rustem, B., Howe, M. 1999. A quasi-Newton algorithm for continuous minimax. IC-DoC Research Report.

Rustem, B., Nguyen, Q., 1998. An algorithm for the inequality constrained discrete min–max problem. SIAM Journal Optimization 8, 265–283.

Shanno, D.F., 1970. Conditioning of Quasi-Newton Methods for Function Minimization. Mathematics of Computation 24, 647–654.