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Rep-tiles with woven horns

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Abstract

This paper describes a simple geometric construction for the visualization of Alexander's horned sphere as a self-similar fractal curve in the plane. The construction is based on a recursive rep-2 rectangle progression to a specified depth. Parameterized curve generation and rendering details are briefly discussed.

Keywords: Alexander's horned sphere; Rep-tile; Rep-2 rectangle; Fractal; Art

1. Introduction

Alexander's horned sphere is an interesting topological structure first introduced by Alexander, 1924 [1]. It can be informally described as a horn shape with recursively bifurcating interlocked tips that approach each other with increasing precision but never actually meet.

The horned sphere is homeomorphic with the ball $\boldsymbol{B}^3$, has a sphere as its boundary, and is a wild embedding in $\boldsymbol{E}^3$. Although composed of a countable union of compact sets, the horned sphere has an uncountable infinity of wild points that are the limits of the sequences of the branch points (the "ends" of the horns) [2].

The horned sphere is traditionally visualized as a recursive set of orthogonally interlocked ring pairs of decreasing radius [3]. However, it can be embedded in the plane by reducing the interlock angle between ring pairs from 90° to 0°, then applying a weave pattern to reestablish ring interlock without self-intersection (Fig. 1). This figure was inspired by the etching "Yggdrasil" by Bill Meyers [4].

It should be noted that the woven horn pair shown in Fig. 1 is incomplete for aesthetic purposes. To be topologically equivalent to the horned sphere its roots must be extended to meet, giving a single woven shape rather than two woven halves.

2. Rep-tiles

A rep-tile (short for *repetitive tiling*) is a polygon that tiles a larger version of itself using identical copies of itself [5]. A polygon that can be divided into k congruent copies of itself is described as a rep-k polygon.

The rep-2 rectangle with sides in the ratio $1{:}\sqrt{2}$ is of particular interest. This is the ratio used for metric paper sizes A2, A3, A4, etc. ; each page can be folded in half to give two pages of the next smaller size, simplifying stacking and printing tasks [6].

Rep-tiles provide a convenient method for tiling an area to a given depth of recursion. For instance, Fig. 2 shows a $1{:}\sqrt{2}$ rectangle progression to three levels of subdivision; note that the side length ratio is preserved with each iteration. This fractal structure forms a rectangular lattice upon which the planar woven horn can be generated.

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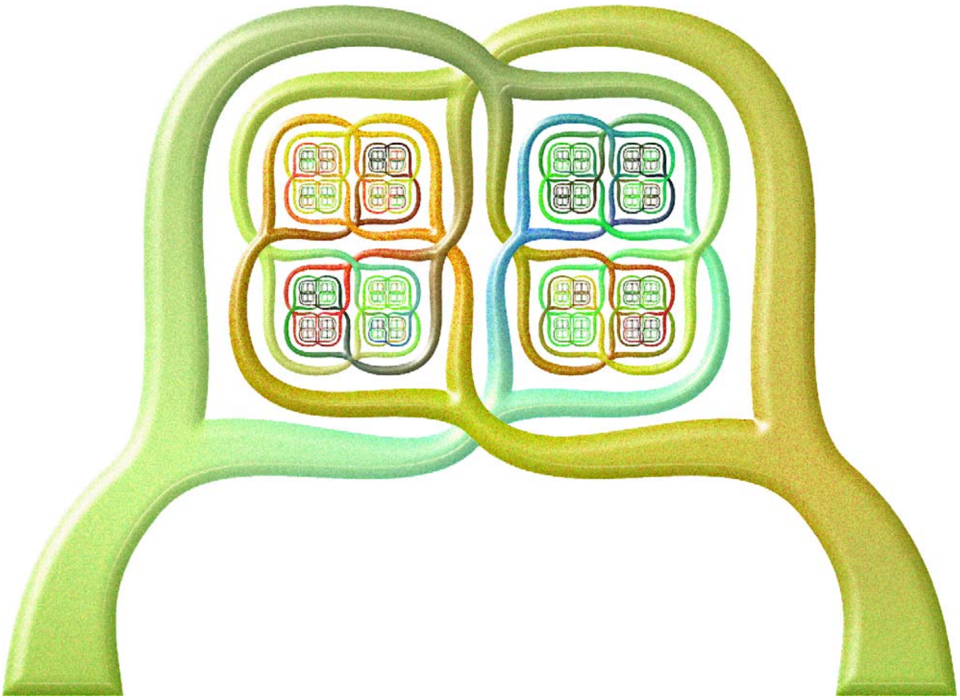

Fig. 1. A pair of woven horns.

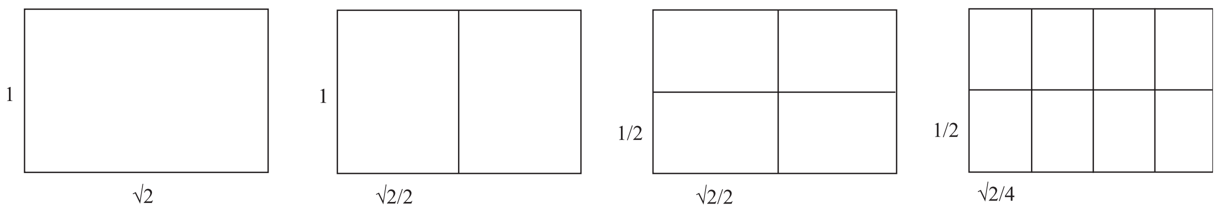

Fig. 2. Rep-2 rectangle progression.

3. Curve geometry

The construction of one iteration of horn growth, from a pair of left and right parent branches P_l and P_r, is shown in Fig. 3. The generated cubic spline director curves are bounded by the $1{:}\sqrt{2}$ rectangle with width w and height h, where:

$w = |P_l - P_r|$, the gap distance between the terminal points of P_l and P_r

$h = w/\sqrt{2}$.

The interior arrows indicate the direction and origin of the next generation of horn growth. Child horns are rotated 90° relative to their parents and bounded by smaller $1{:}\sqrt{2}$ subdivisions (shaded). These subrectangles are further reduced in size by a spacing parameter s

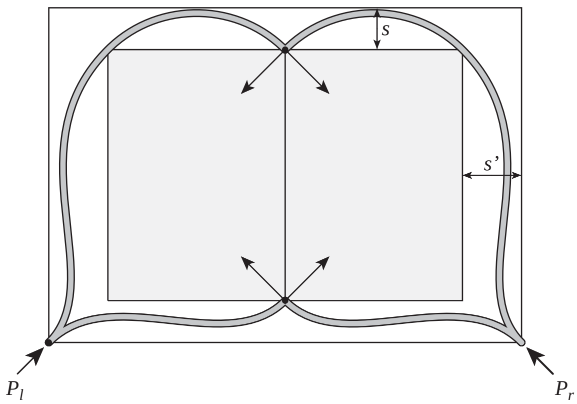

Fig. 3. One iteration of recursive horn growth.

($s' = \sqrt{2}\,s$), accelerating each branch's attraction towards a central vanishing point.

Outline curves are obtained by offsetting the director curves as shown in Fig. 4 (left). The offset radius tapers linearly from each branch's origin to its tip. The two intersecting outlines of each horn pair are clipped at their intersection point x giving the final outlines shown on the right. Each branch's base point is marked as either an overpass or an underpass, and further intersection testing and clipping is performed to introduce the weave pattern.

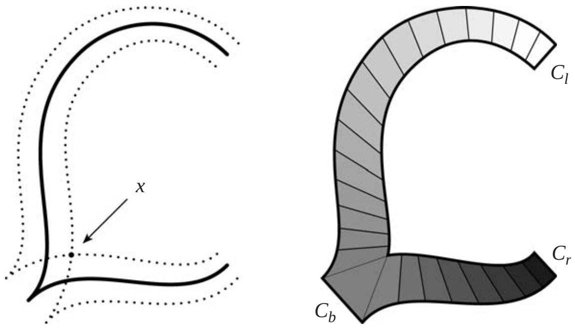

Fig. 4. Geometric construction of horn curves.

The resulting outline pairs are then sampled at regular intervals to provide a number of slices. Each slice is shaded a color interpolated between the horn's base color C_b and the color at the respective tip, C_l or C_r.

The reduction factor for each iteration is given by $(1-2s)/\sqrt{2}$, hence the *self-similarity dimension* (D_s) for the woven horn, excluding the root trunks, is given by

$$D_s = \frac{\log(\text{number of self}-\text{similar pieces})}{\log(1/\text{reduction factor})} = \frac{\log(2)}{\log\left(1/\left((1-2s)/\sqrt{2}\right)\right)}.$$

As s approaches 0, then the self-similarity dimension approaches 2:

$$D_s = \frac{\log(2)}{\log\left(1/\left(1/\sqrt{2}\right)\right)} = \frac{\log(2)}{\log(\sqrt{2})} = 2.$$

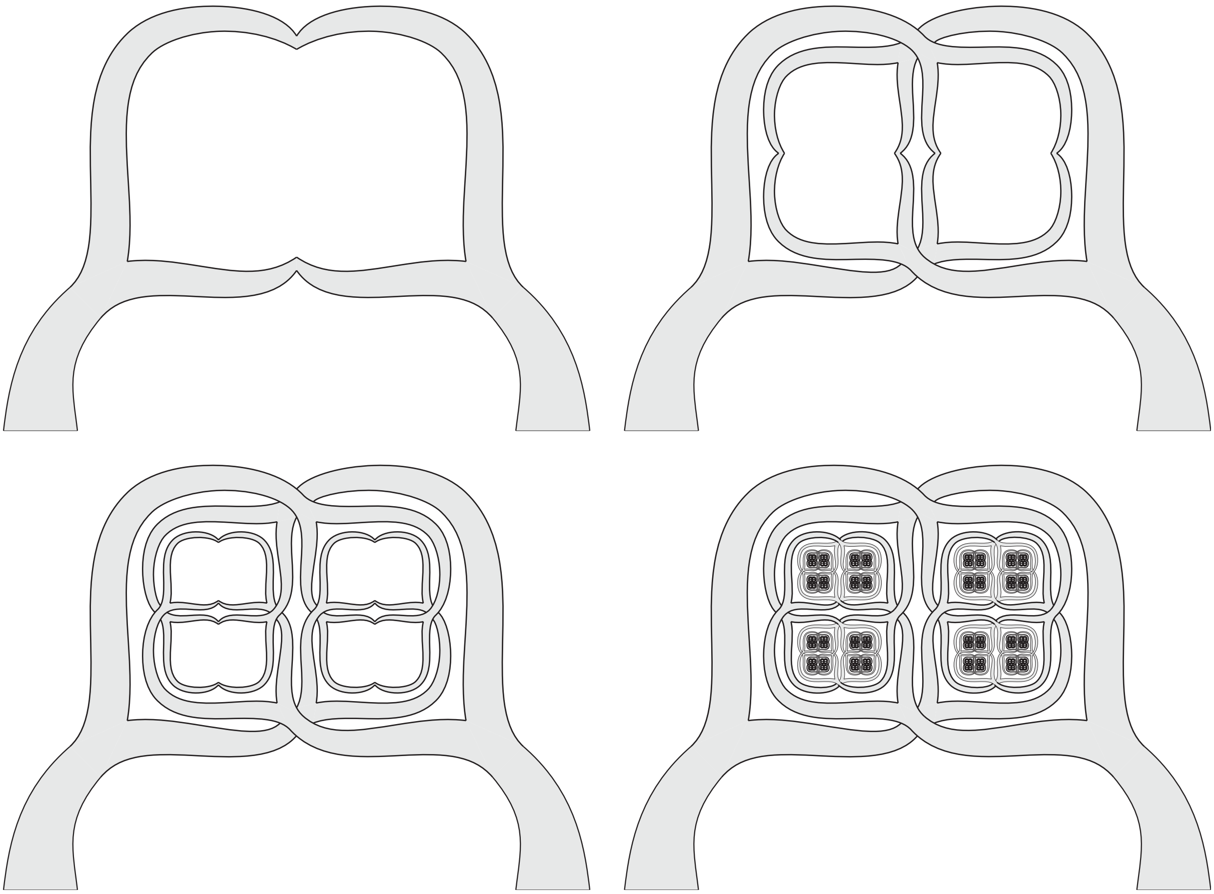

Fig. 5. Increasing iteration depth: $d = 1$, $d = 2$, $d = 3$, and $d = 9$.

4. Examples

Fig. 5 illustrates the process of iterative horn growth to various levels of recursion. The gap width w_d of the branching pair at any particular depth d can be calculated from the spacing s and the total structure width W as follows:

$$w_d = w_{d-1}/\sqrt{2} - 2s_{d-1},$$

where $w_0 = W$ and s_d is the spacing for iteration d.

The spacing parameter s governs the weave density of the resulting horn, as shown in Fig. 6. Smaller values yield boxier shapes with a denser weave around the perimeter of each rep-tile, converging to the underlying rectangular lattice as s approaches 0. Larger spacing values provide a rounder, looser weaves with a tighter central cluster. The optimal spacing for aesthetic purposes was found to be around $s = 13\%$.

The figures for this paper were created using a Visual C++ program that takes as input a number of relevant parameters including:

Depth	Depth of recursion
Spacing	Spacing parameter s as a percentage of width
Trunk thickness	Initial trunk thickness governing total thickness of horn set
Branch thinning	Attenuation of branch thickness with each generation
Resolution	Desired level of curve subdivision for output splines

The program performs the appropriate curve generation, intersection and clipping, and provides a preview of

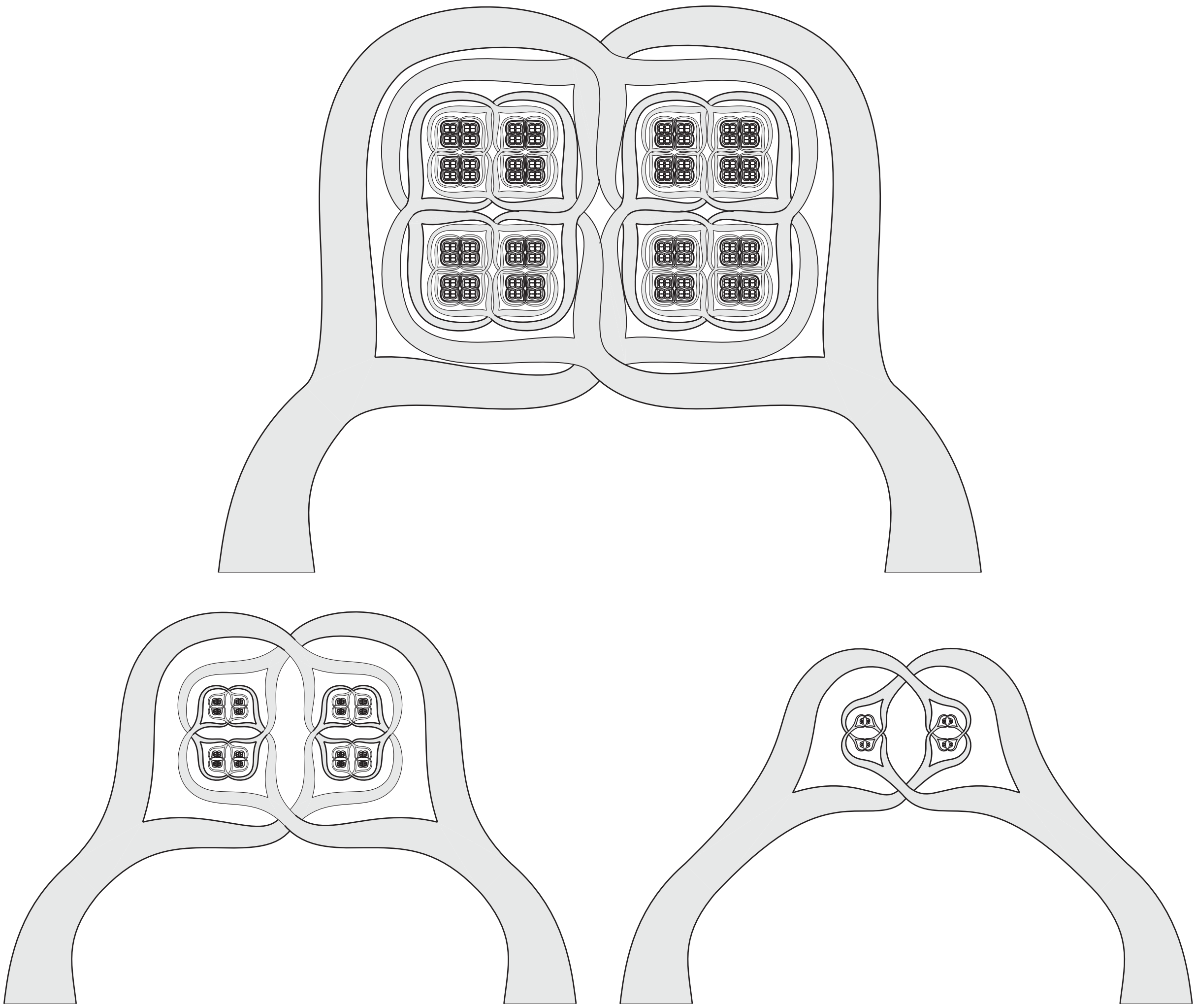

Fig. 6. Spacing as a percentage of width: $s = 10\%$, $s = 20\%$, and $s = 30\%$.

the resulting horn for each parameter adjustment. It produces as output an Adobe Postscript file containing outline curves in cubic spline format and fill information for the current curve set. The color image (Fig. 1) was imported into Adobe Photoshop and further enhanced with the Eye Candy 4000 filters "Bevel Boss", "HSB Noise" and "Glass".

5. Conclusion

This paper demonstrates the use of a 1:$\sqrt{2}$ rectangle progression as a frame of reference upon which the woven horn can be hung. This complex construct becomes simple when broken down into its generational components, and characteristics of the resulting design can be conveniently parameterized.

Future work might involve the use of different rep-tiles (or even irrep-tiles) as suggested by Paul van Wamelen, and the introduction of other forms of noise to produce a less ordered and more organic-looking result.

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